# ON POLYNOMIALS WITH SPAN LESS THAN 4

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Abstract. Various results concerning polynomials with span less than 4 are given.

## Introduction

We consider monic integral irreducible polynomials whose zeros belong to a real interval $I$ of length less than 4.

## Kronecker-polynomials

Here $I = (-2, 2)$.

We can represent the polynomial in the form

$$P(x) = \prod_{j=1}^{n} (x - 2\cos v_j), \tag{1}$$

where we can stipulate that $0 < v_j < \pi$ for all $j$.

A new polynomial related to $P$ is defined by

$$Q(t) = t^n P(t + \frac{1}{t}) = \prod_{j=1}^{n} (t^2 - 2t\cos v_j + 1). \tag{2}$$

Evidently, the zeros of $Q$ are $e^{\pm i v_j}$.

We also have the representation

$$P(x) = \sum_{j=0}^{n} b_j x^j, \tag{3}$$

with integral coefficients and with $b_n = 1$.

Thus

$$Q(t) = \sum_{j=0}^{n} b_j t^{n-j} (t^2 + 1)^j, \tag{4}$$

which shows that $Q$ is integral and monic too. Since its zeros belong to the unit circle, they must be roots of unity.

If $Q$ (and so the corresponding $P$) is irreducible, the degree $2n$ has the form $\phi(m)$ for some $m > 2$, and the parameters $v_j$ have the form

$$v_j = 2\pi \frac{k_j}{m}, \tag{5}$$

where the integers $k_j$ run through the integers in the interval $(0, \frac{m}{2})$ prime to $m$. In this case $Q$ divides the polynomial $(t^{2m} - 1)/(t^2 - 1)$. Putting $t = e^{iv}$ we see that this last polynomial in $t$ corresponds to the function $(\sin mv)/\sin v$, regarded as a polynomial in $2\cos v$.

## Non-Kronecker-polynomials

We use this name when the interval $I$ cannot be chosen such that its endpoints are integral.

In [1] Robinson conducted what seems to be an exhaustive search for polynomials of this type of degree at most 8. I used various criteria to find more polynomials of higher degree. I re-found Robinson's 21 polynomials of degree 8. I found 19 polynomials of degree 9, and the number of polynomials of a certain degree found by me tended to decrease with increasing degree, but not in a regular way. I found, for instance, 9 polynomials of degree 12, 4 of degree 13, and again 9 of degree 14. As my search cannot be guaranteed to be exhaustive, no conclusions can be drawn with respect to the real number of polynomials of a given degree. I found no non-Kronecker polynomials of degree higher than 18.

More details with respect to the numerical results are given in the next section.

I found a weak connection between polynomials corresponding to the two intervals $I_1 = (-\frac{9}{4}, \frac{7}{4})$ and $I_2 = (-\frac{5}{2}, \frac{3}{2})$. In fact, if $Q$ has its zeros in $I_1$, then $Q \circ P$, where $P(x) = x^2 + x - 2$, has its zeros in $I_2$. However, it does not follow from irreducibility of $Q$ that $Q \circ P$ is also irreducible, so a supplementary check is necessary. Among the examples given in the next section the polynomials of the type $Q \circ P$ (16c, 14g, 14h, 14i, 12i, 10m, 10n, 10o, 10p) are easily recognizable by the fact that if $\xi$ is a zero, then $-1 - \xi$ is also. Robinson's polynomials 4n, 6h, 8a, and 8c are also of this type, but normalized differently.

One of the criteria I used in my search for monic integral polynomials with all zeros in a given interval $I = (c, c+4)$ was minimality of norm, since this presumably could lead to maximality of number of real zeros in $I$. As an example, take the weight function $w(x) = (x-c)^\alpha (c+4-x)^\beta$. The corresponding orthogonal system consists of suitably modified Jacobi polynomials $S_n(x) = \frac{1}{k_n} P_n^{(\alpha,\beta)}(\frac{x-c-2}{2})$, where the factor $k_n$ ensures that $S_n$ is monic.

Expanding the powers $x^n = \sum_{j=0}^n a_{n,j} S_j(x)$, where the coefficients $a_{n,j}$ are easily found by a recursion related to the one valid for the Jacobi polynomials, the problem is to minimize

$$\begin{aligned} \int_c^{c+4} dx\, w(x)(\sum_{j=0}^n b_j x^j)^2) &= \int_c^{c+4} dx\, w(x)(\sum_{j=0}^n S_j(x) \sum_{k=j}^n b_k a_{k,j})^2 \\ &= \sum_{j=0}^n (\int_c^{c+4} dx\, w(x)(S_j(x))^2)(\sum_{k=j}^n b_k a_{k,j})^2, \end{aligned} \tag{6}$$

with $b_n = 1$ and the other coefficients integral.

I approximate the solution by successively determining the integers $b_{n-1}, \ldots, b_0$ as close to the solution of the equations $\sum_{j=k}^n b_j a_{j,k} = 0$ (where $k = n-1, \ldots, 0$) as possible. Obviously, this does, in general, give a more satisfactory result than just rounding off each coefficient in $S_n(x)$.

This approach was varied by choosing weight functions obtained from the one above by multiplying it with polynomials of low degree non-negative in $I$.

**Numerical results.**

The polynomials were found in 1996 by means of a Visual Basic programme using the methods described above. For each polynomial the zeros were checked using a Derive programme.

The parameter $c$ was chosen to satisfy $-\frac{5}{2} < c < -2$.

*18a.*

$$P(x) = x^{18} + 2x^{17} - 16x^{16} - 31x^{15} + 107x^{14} + 198x^{13} - 388x^{12} - 672x^{11} + 827x^{10} + 1302x^9 - 1048x^8 - 1436x^7 + 758x^6 + 844x^5 - 280x^4 - 225x^3 + 40x^2 + 19x - 1$$

with zeros

$-2.0663652530, -2.0226196327, -1.8626294218, -1.7528461389, -1.5744074455, -1.1951384168, -1.0697337215, -0.5973210154, -0.3332193541, 0.0490376194, 0.4266797804, 0.7232891540, 1.0441059232, 1.3516003906, 1.4553286687, 1.6782063642, 1.8363562638, 1.9096762354.$

*18b.*

$$P(x) = x^{18} + 6x^{17} - x^{16} - 66x^{15} - 68x^{14} + 293x^{13} + 444x^{12} - 678x^{11} - 1242x^{10} + 891x^9 + 1816x^8 - 695x^7 - 1396x^6 + 333x^{+}500x^4 - 89x^3 - 56x^2 + 5x + 1$$

with zeros

$-2.2865697074, -2.2078844523, -2.1368991528, -1.9005931557, -1.8364741779, -1.5760189020, -1.2254104021, -0.9617825877, -0.3424797834, -0.1063052903, 0.1886387956, 0.4856047961, 0.7946796187, 1.0456051331, 1.2934470620, 1.4218876563, 1.6402760393, 1.7102785106.$

*18c.*

$$P(x) = x^{18} + 8x^{17} + 12x^{16} - 58x^{15} - 174x^{14} + 116x^{13} + 770x^{12} + 108x^{11} - 1702x^{10} - 734x^9 + 2095x^8 + 1065x^7 - 1440x^6 - 637x^5 + 496x^4 + 138x^3 - 56x^2 - 8x + 1$$

with zeros

$-2.4071539843, -2.3321397832, -2.2684191710, -2.0938230590, -1.9039167355, -1.6388535778, -1.3554330145, -1.1250823061, -0.4344102865, -0.2117384004, 0.0866800366, 0.3633157681, 0.7583264971, 0.9480591325, 1.1658443448, 1.3898925141, 1.4704775675, 1.5883744578.$

*17a.*

$$P(x) = x^{17} + 2x^{16} - 15x^{15} - 30x^{14} + 91x^{13} + 183x^{12} - 285x^{11} - 583x^{10} + 485x^9 + 1036x^8 - 424x^7 - 1014x^6 + 142x^5 + 500x^4 + 20x^3 - 96x^2 - 15x + 1$$

with zeros

$-2.0730958018, -2.0153135003, -1.8203286845, -1.6455788197, -1.3497380723, -1.2416642005, -0.8209073249, -0.6405636145, -0.2533561355, 0.0506454257, 0.6882447122, 0.9514039364, 1.1892660090, 1.4632493818, 1.7221368514, 1.8711131171, 1.9244867207.$

*17b.*

$$P(x) = x^{17} + 4x^{16} - 9x^{15} - 50x^{14} + 22x^{13} + 254x^{12} + 26x^{11} - 676x^{10} - 208x^9 + 1012x^8 + 361x^7 - 846x^6 - 259x^5 + 362x^4 + 66x^3 - 61x^2 - x + 1$$

with zeros

$-2.1631304658, -2.1322007273, -1.9642406922, -1.7536139460, -1.5304145313, -1.3246687104, -1.0810258405, -0.6763649307, -0.1341781960, 0.1367879721, 0.4549134019, 0.8061887852, 1.0782301804, 1.2649961714, 1.472099603, 1.7211413389, 1.8254805872.$

*17c.*

$$P(x) = x^{17} + 6x^{16} - 58x^{14} - 58x^{13} + 232x^{12} + 306x^{11} - 504x^{10} - 694x^9 + 655x^8 + 789x^7 - 515x^6 - 431x^5 + 220x^4 + 92x^3 - 35x^2 - 6x + 1$$

with zeros

$-2.4212210628, -2.3510318431, -2.2293150199, -1.9888247580, -1.7232196696, -1.2736952678, -1.0964929643, -0.4947053245, -0.2493831084, 0.1166771043, 0.4874982523, 0.6601857564, 0.8985750601, 1.2232221973, 1.3959310990, 1.4673723325, 1.5784272166.$

*16a.*

$$P(x) = x^{16} + x^{15} - 16x^{14} - 13x^{13} + 106x^{12} + 66x^{11} - 373x^{10} - 164x^9 + 742x^8 + 203x^7 - 818x^6 - 109x^5 + 451x^4 + 10x^3 - 93x^2 + 6x + 1$$

with zeros

$-2.1274246129, -2.0131792908, -1.8631064444, -1.7575934255, -1.2908875551, -1.0352263685, -0.6864149289, -0.0768989917, 0.1533063741, 0.6110497734, 1.1183824868, 1.2036739164, 1.4389499381, 1.6482502251, 1.8129883572, 1.8641305468.$

*16c.*

$$P(x) = x^{16} + 8x^{15} + 13x^{14} - 49x^{13} - 154x^{12} + 77x^{11} + 583x^{10} + 88x^9 - 1104x^8 - 379x^7 + 1151x^6 + 380x^5 - 642x^4 - 129x^3 + 158x^2 + 10x - 11$$

with zeros

$-2.4983530167, -2.4381853818, -2.2633256806, -2.0834955555, -1.8494124928, -1.5437805085, -1.3332992724, -0.6687231828, -0.3312768172, 0.3332992725, 0.5437805085, 0.8494124929, 1.0834955555, 1.2633256806, 1.4381853818, 1.4983530167.$

*15a.*

$$\begin{aligned}P(x) =& x^{15} + x^{14} - 15x^{13} - 12x^{12} + 91x^{11} + 54x^{10} - 283x^9 \\ & - 111x^8 + 470x^7 + 100x^6 - 393x^5 - 30x^4 + 140x^3 - 15x + 1\end{aligned}$$

with zeros

$-2.1076916949, -2.0351459328, -1.9280936579, -1.7152990576, -1.1907746020,$
$-0.6943077224, -0.4577761601, 0.0697433820, 0.3807578774, 0.7854804301,$
$1.0979141886, 1.4182363821, 1.6694752110, 1.8152201090, 1.8922612475.$

*15b.*

$$\begin{aligned}P(x) =& x^{15} + x^{14} - 15x^{13} - 12x^{12} + 92x^{11} + 55x^{10} - 294x^9 \\ & - 119x^8 + 515x^7 + 120x^6 - 474x^5 - 45x^4 + 196x^3 + x^2 - 22x - 1\end{aligned}$$

with zeros

$-2.1172121279, -2.0196516213, -1.8912746122, -1.6305726750, -1.2165658513,$
$-0.9621801352, -0.3548501320, -0.0462430955, 0.4957667807, 0.8758116954,$
$1.2261715247, 1.3864332458, 1.6334233440, 1.7528514878, 1.8680921719.$

*15c.*

$$\begin{aligned}P(x) =& x^{15} + 3x^{14} - 11x^{13} - 37x^{12} + 45x^{11} + 181x^{10} - 79x^9 \\ & - 445x^8 + 34x^7 + 572x^6 + 67x^5 - 357x^4 - 84x^3 + 81x^2 + 27x + 1\end{aligned}$$

with zeros

$-2.1420811705, -2.0504663181, -1.8854164232, -1.5939789914, -1.4780521141,$
$-1.0887156634, -0.6544242185, -0.320298018, -0.0427067600, 0.7759169473,$
$1.0517428925, 1.2969899523, 1.5526831051, 1.7461630052, 1.8326437749.$

*15d.*

$$\begin{aligned}P(x) =& x^{15} + 2x^{14} - 13x^{13} - 24x^{12} + 69x^{11} + 113x^{10} - 191x^9 \\ & - 263x^8 + 291x^7 + 312x^6 - 235x^5 - 173x^4 + 87x^3 + 34x^2 - 10x - 1\end{aligned}$$

with zeros

$-2.2158264280, -2.1019564143, -1.8168330084, -1.5176631110, -1.3117118339,$
$-0.9396808301, -0.5109358819, -0.0824604009, 0.3262703351, 0.6512931220,$
$1.0715792078, 1.3374057486, 1.5814735899, 1.7478891101, 1.7811567948.$

*15e.*

$$\begin{aligned}P(x) =& x^{15} + 6x^{14} + x^{13} - 54x^{12} - 66x^{11} + 188x^{10} + 325x^9 \\ & - 313x^8 - 697x^7 + 238x^6 + 747x^5 - 38x^4 - 382x^3 - 44x^2 + 72x + 17\end{aligned}$$

with zeros

$-2.3840953000, -2.3239498573, -2.0877025066, -1.8904125474, -1.5974114549,$
$-1.2915176919, -0.9113532104, -0.5361516753, -0.2970941374, 0.7066901367,$
$0.9268688412, 1.1694146702, 1.3676972611, 1.5400341554, 1.6089833166.$

*15f.*

$$\begin{aligned}P(x) =& x^{15} + 7x^{14} + 7x^{13} - 50x^{12} - 103x^{11} + 127x^{10} + 393x^9 \\ & - 129x^8 - 692x^7 + 35x^6 + 597x^5 + x^4 - 217x^3 + 10x^2 + 15x - 1\end{aligned}$$

with zeros

$-2.4550168166, -2.3977853093, -2.2572610395, -1.9508968185, -1.6651377208,$
$-1.5465025161, -0.9783431165, -0.3007552615, 0.0680822846, 0.2920439402,$
$0.8181097935, 1.0520580932, 1.3446225747, 1.4352415083, 1.5415404044.$

*14a.*

$$\begin{aligned}P(x) =& x^{14} + 3x^{13} - 10x^{12} - 33x^{11} + 38x^{10} + 141x^9 \\ & - 68x^8 - 293x^7 + 56x^6 + 301x^5 - 15x^4 - 135x^3 - x^2 + 17x - 1\end{aligned}$$

with zeros

$-2.1808014622, -2.1138295111, -1.9531798694, -1.6279282050, -1.2304896945,$
$-0.8979744292, -0.5352283100, 0.0608254510, 0.3689514901, 0.9450109805,$
$1.2647115267, 1.4437306622, 1.6431080129, 1.8130933580.$

*14b.*

$$\begin{aligned}P(x) =& x^{14} + 4x^{13} - 7x^{12} - 42x^{11} + 8x^{10} + 171x^9 + 46x^8 \\ & - 337x^7 - 145x^6 + 324x^5 + 143x^4 - 131x^3 - 43x^2 + 14x + 1\end{aligned}$$

with zeros

$-2.2164817836, -2.1452838407, -1.9167382105, -1.6848475652, -1.5255194601,$
$-1.0582638856, -0.5672001904, -0.0619852121, 0.2798971805, 0.7656278945,$
$1.2714824107, 1.4570182424, 1.6387897044, 1.7635047158.$

*14c.*

$$\begin{aligned}P(x) =& x^{14} + 4x^{13} - 6x^{12} - 38x^{11} + 3x^{10} + 137x^9 + 44x^8 \\ & - 234x^7 - 103x^6 + 193x^5 + 82x^4 - 69x^3 - 22x^2 + 8x + 1\end{aligned}$$

with zeros

$-2.2273955845, -2.1209571856, -1.9365037746, -1.7148702425, -1.2502374293,$
$-0.8485632620, -0.5924527665, -0.1054688001, 0.3571783248, 0.6791474101,$
$1.0583144506, 1.2812989444, 1.6676670601, 1.7528428551.$

*14d.*

$$\begin{aligned}P(x) =& x^{14} + 3x^{13} - 11x^{12} - 35x^{11} + 49x^{10} + 162x^9 - 114x^8 \\ & - 375x^7 + 147x^6 + 448x^5 - 99x^4 - 253x^3 + 26x^2 + 49x + 1\end{aligned}$$

with zeros

$-2.2559169575, -2.1983207689, -2.0199331390, -1.6297198125, -1.2546374271,$
$-1.0599781419, -0.5304886268, -0.0206803567, 0.6912833265, 1.0710728311,$
$1.3148727858, 1.4984643436, 1.6540891507, 1.7398927928.$

*14e.*

$$\begin{aligned}P(x) =& x^{14} + 6x^{13} + 3x^{12} - 42x^{11} - 59x^{10} + 109x^9 + 206x^8 \\ & - 131x^7 - 308x^6 + 77x^5 + 214x^4 - 23x^3 - 62x^2 + 3x + 5\end{aligned}$$

with zeros

$-2.4001871560, -2.2961656545, -2.0648683053, -1.8320888119, -1.4522711670,$
$-1.1105074859, -0.6969470241, -0.3301630483, 0.3816235147, 0.6982431701,$
$0.8999527483, 1.1830284587, 1.4576278703, 1.5627228911.$

*14f.*

$$\begin{aligned}P(x) =& x^{14} + 6x^{13} + 2x^{12} - 46x^{11} - 54x^{10} + 140x^9 + 198x^8 \\ & - 223x^7 - 295x^6 + 203x^5 + 180x^4 - 93x^3 - 30x^2 + 9x + 1\end{aligned}$$

with zeros

$-2.4405942387, -2.3584399425, -2.2146923803, -1.9575792128, -1.6457763390,$
$-1.1376544798, -0.3840252036, -0.0921617733, 0.3166710281, 0.7396291560,$
$0.9078904853, 1.2578687494, 1.4543672742, 1.5544968770.$

*14g.*

$$\begin{aligned}P(x) =& x^{14} + 7x^{13} + 8x^{12} - 43x^{11} - 94x^{10} + 91x^9 + 308x^8 \\ & - 73x^7 - 465x^6 + 16x^5 + 338x^4 - 6x^3 - 100x^2 + 4x + 7\end{aligned}$$

with zeros

$-2.4931067392, -2.3816834071, -2.2271390536, -1.9214970810, -1.7023414401,$
$-1.3598481572, -0.7194675990, -0.2805324010, 0.3598481573, 0.7023414402,$
$0.9214970810, 1.2271390536, 1.3816834071, 1.4931067392.$

*14h.*

$$\begin{aligned}P(x) =& x^{14} + 7x^{13} + 8x^{12} - 43x^{11} - 95x^{10} + 86x^9 + 308x^8 \\ & - 43x^7 - 445x^6 - 50x^5 + 291x^4 + 55x^3 - 68x^2 - 12x + 1\end{aligned}$$

with zeros

$-2.4988957488, -2.4083058129, -2.2410001876, -2.0439443443, -1.6480775764,$
$-1.06261074462, -0.7516464705, -0.2483535295, 0.0626107446, 0.6480775764,$
$1.0439443443, 1.2410001876, 1.4083058129, 1.4988957488.$

*14i.*

$$\begin{aligned}P(x) =& x^{14} + 7x^{13} + 9x^{12} - 37x^{11} - 90x^{10} + 56x^9 + 258x^8 \\ & + 3x^7 - 332x^6 - 64x^5 + 197x^4 + 37x^3 - 44x^2 - 3x + 1\end{aligned}$$

with zeros

$-2.4997490549, -2.3345329405, -2.1612065838, -1.9249528984, -1.5495945244,$
$-1.1299998895, -0.8112009050, -0.1887990950, 0.1299998896, 0.5495945244,$
$0.9249528985, 1.1612065838, 1.3345329405, 1.4997490549.$

*13a.*

$$\begin{aligned}P(x) =& x^{13} + 2x^{12} - 11x^{11} - 21x^{10} + 46x^9 + 82x^8 \\ & - 90x^7 - 145x^6 + 81x^5 + 111x^4 - 28x^3 - 27x^2 + 3x + 1\end{aligned}$$

with zeros

$-2.1654584104, -2.0187961023, -1.7684382080, -1.3854946251, -1.0923917482,$
$-0.5800100639, -0.1606961233, 0.2619507350, 0.5920316670, 1.2107303084,$
$1.5149396232, 1.7584987259, 1.8331342218.$

*13b.*

$$\begin{aligned}P(x) =& x^{13} + 4x^{12} - 5x^{11} - 33x^{10} + 2x^9 + 103x^8 \\ & + 24x^7 - 150x^6 - 42x^5 + 101x^4 + 22x^3 - 26x^2 - 3x + 1\end{aligned}$$

with zeros

$-2.3114404819, -2.2114239521, -1.9785133364, -1.4426589075, -1.1481309011,$
$-0.7518533050, -0.2725794462, 0.1604743028, 0.6517655220, 0.8909652860,$
$1.2540837330, 1.5269311700, 1.6323803163.$

*13c.*

$$\begin{aligned}P(x) =& x^{13} + 6x^{12} + 3x^{11} - 42x^{10} - 60x^9 + 104x^8 \\ & + 205x^7 - 104x^6 - 283x^5 + 28x^4 + 159x^3 + 10x^2 - 25x - 1\end{aligned}$$

with zeros

$-2.3949166621, -2.3214268352, -2.1110078675, -1.8261866042, -1.5268927413,$
$-0.9249300066, -0.6291032704, -0.0397635008, 0.4440280494, 1.0372595113,$
$1.1977468465, 1.5158549080, 1.5793381730.$

*13d.*

$$\begin{aligned}P(x) =& x^{13} + 6x^{12} + 3x^{11} - 41x^{10} - 55x^9 + 105x^8 \\ & + 178x^7 - 129x^6 - 234x^5 + 83x^4 + 126x^3 - 26x^2 - 20x + 2\end{aligned}$$

with zeros

$-2.4255192844, -2.3273353880, -2.2006879319, -1.8493102526, -1.5103260918,$
$-0.9569037399, -0.4695267014, 0.0939814504, 0.5756698576, 0.9105271383,$
$1.1543685579, 1.4629666463, 1.5420957397.$

*12a.*

$$\begin{aligned}P(x) =& x^{12} + 2x^{11} - 11x^{10} - 21x^9 + 46x^8 + 82x^7 \\ & - 90x^6 - 146x^5 + 80x^4 + 116x^3 - 24x^2 - 33x - 1\end{aligned}$$

with zeros

$-2.1173177833, -2.0594315249, -1.7954596772, -1.4416342991, -0.8846695715,$
$-0.7162027189, -0.0311103700, 0.8414127102, 1.0879225978, 1.5360762223,$
$1.6996448714, 1.8807695432.$

*12b.*

$$\begin{aligned}P(x) =& x^{12} + 2x^{11} - 10x^{10} - 19x^9 + 37x^8 + 65x^7 \\ & - 61x^6 - 96x^5 + 41x^4 + 57x^3 - 6x^2 - 9x - 1\end{aligned}$$

with zeros

$-2.1289725920, -2.0350468729, -1.7703226436, -1.2651471454, -0.8746623806, -0.3374175584, -0.1386157658, 0.5414088867, 1.0534967650, 1.3983939533, 1.7109157543, 1.8459695993.$

*12c.*

$$\begin{aligned}P(x) =& x^{12} + 2x^{11} - 11x^{10} - 20x^9 + 48x^8 + 74x^7 \\ & - 105x^6 - 123x^5 + 118x^4 + 88x^3 - 61x^2 - 21x + 11\end{aligned}$$

with zeros

$-2.1669634578, -2.0497340692, -1.8109307602, -1.5423025253, -0.9411818223, -0.6644421341, 0.3970550898, 0.7300783496, 1.1120067610, 1.4464894094, 1.6815624643, 1.8083626949.$

*12d.*

$$\begin{aligned}P(x) =& x^{12} + 2x^{11} - 10x^{10} - 18x^9 + 39x^8 + 58x^7 \\ & - 74x^6 - 79x^5 + 67x^4 + 40x^3 - 22x^2 - 4x + 1\end{aligned}$$

with zeros

$-2.1883557950, -2.0250378384, -1.7451195659, -1.4620734950, -0.8447941278, -0.2822241344, 0.1593298925, 0.5486344907, 1.1096036542, 1.3202481170, 1.6051425454, 1.8046462568.$

*12e.*

$$\begin{aligned}P(x) =& x^{12} + 3x^{11} - 8x^{10} - 26x^9 + 25x^8 + 83x^7 \\ & - 41x^6 - 117x^5 + 39x^4 + 67x^3 - 18x^2 - 10x + 1\end{aligned}$$

with zeros

$-2.2781018907, -2.1900684386, -1.8349133430, -1.4664143619, -1.0502483662, -0.3839473399, 0.0904241228, 0.6769109576, 0.8676712240, 1.3247664597, 1.5499166878, 1.6940042885.$

*12f.*

$$\begin{aligned}P(x) =& x^{12} + 4x^{11} - 5x^{10} - 33x^9 + x^8 + 100x^7 \\ & + 29x^6 - 132x^5 - 49x^4 + 67x^3 + 23x^2 - 6x - 1\end{aligned}$$

with zeros

$-2.3267418529, -2.2182570180, -2.0354974150, -1.4704694221, -1.0766243115, -0.5186242043, -0.1285051053, 0.2736857090, 0.9607836233, 1.3496499677, 1.5407783667, 1.6498216627.$

*12g.*

$$\begin{aligned} P(x) =& x^{12} + 3x^{11} - 8x^{10} - 26x^9 + 25x^8 + 84x^7 \\ & - 39x^6 - 122x^5 + 30x^4 + 75x^3 - 8x^2 - 13x - 1 \end{aligned}$$

with zeros

$-2.3505384498, -2.0786770966, -1.9483604278, -1.2866125999, -1.0847064646, -0.3864443445, -0.0846809773, 0.6276925019, 1.1164461744, 1.3201140220, 1.5191206060, 1.6366470561.$

*12h.*

$$\begin{aligned} P(x) =& x^{12} + 5x^{11} - 32x^9 - 27x^8 + 71x^7 \\ & + 77x^6 - 65x^5 - 71x^4 + 25x^3 + 20x^2 - 4x - 1 \end{aligned}$$

with zeros

$-2.3756730323, -2.2259720131, -2.0437136487, -1.4339262345, -1.1653877036, -0.6688170751, -0.1585496936, 0.3711895489, 0.5663680428, 1.0363807250, 1.4943745981, 1.6037264862.$

*12i.*

$$\begin{aligned} P(x) =& x^{12} + 6x^{11} + 5x^{10} - 30x^9 - 49x^8 + 50x^7 \\ & + 112x^6 - 31x^5 - 99x^4 + 5x^3 + 30x^2 - 1 \end{aligned}$$

with zeros

$-2.4745859916, -2.2678878619, -2.0722573635, -1.6975712777, -1.1914941269, -0.8020734334, -0.1979265666, 0.1914941269, 0.6975712778, 1.0722573635, 1.2678878619, 1.4745859916.$

*11a.*

$$\begin{aligned} & P(x) = \\ & x^{11} - 11x^9 + x^8 + 44x^7 - 7x^6 - 77x^5 + 15x^4 + 55x^3 - 10x^2 - 11x + 1 \end{aligned}$$

with zeros

$-2.0787261027, -1.8522323437, -1.5194934410, -1.0552856142, -0.5028164122, 0.0873488319, 0.6593212322, 1.1497065508, 1.4796292734, 1.7122165299, 1.9203314958.$

*11b.*

$$\begin{aligned} & P(x) = \\ & x^{11} + x^{10} - 11x^9 - 9x^8 + 45x^7 + 28x^6 - 83x^5 - 35x^4 + 65x^3 + 15x^2 - 15x - 1 \end{aligned}$$

with zeros

$-2.0806824110, -1.9505529193, -1.6732365151, -1.2329343831, -0.6777841469, -0.0637573660, 0.5458837370, 1.0802968163, 1.4490646013, 1.6908436794, 1.9128589074.$

*11c.*

$P(x) =$

$x^{11} + x^{10} - 11x^9 - 9x^8 + 45x^7 + 29x^6 - 83x^5 - 41x^4 + 66x^3 + 25x^2 - 17x - 5$

with zeros

$-2.1361746765, -1.9782442001, -1.5780123747, -1.1163290470, -0.7269602603,$
$-0.2686198972, 0.6742025141, 1.0981113795, 1.4541185843, 1.7154865438,$
$1.8624214342.$

*11d.*

$P(x) =$

$x^{11} + 2x^{10} - 10x^9 - 19x^8 + 37x^7 + 65x^6 - 61x^5 - 95x^4 + 42x^3 + 53x^2 - 9x - 7$

with zeros

$-2.1522801014, -2.0454883060, -1.6556005514, -1.3629416771, -0.9023397506,$
$-0.3584641013, 0.4846571829, 0.9581789007, 1.5411910964, 1.6652037536,$
$1.8278835543.$

*11e.*

$P(x) =$

$x^{11} + 2x^{10} - 9x^9 - 17x^8 + 29x^7 + 50x^6 - 39x^5 - 58x^4 + 19x^3 + 21x^2 - 3x - 1$

with zeros

$-2.1791011167, -1.9700043885, -1.6753539486, -1.1565037357, -0.7528511962,$
$-0.1724167548, 0.3025921917, 0.7175264878, 1.4312215323, 1.6689655883,$
$1.7859253404.$

*11f.*

$P(x) =$

$x^{11} + 2x^{10} - 9x^9 - 16x^8 + 31x^7 + 44x^6 - 50x^5 - 47x^4 + 35x^3 + 15x^2 - 6x - 1$

with zeros

$-2.1835069231, -2.0312062361, -1.6803059323, -1.3485890065, -0.5400533355,$
$-0.1370445131, 0.4333977181, 0.8700016889, 1.3002131454, 1.5660951798,$
$1.7509982145.$

*11g.*

$P(x) =$

$x^{11} + 3x^{10} - 7x^9 - 24x^8 + 17x^7 + 68x^6 - 19x^5 - 81x^4 + 13x^3 + 36x^2 - 5x - 3$

with zeros

$-2.2108964472, -2.0727480449, -1.7790525628, -1.4527034278, -0.9256408874,$
$-0.2491218415, 0.4361904047, 0.8155929055, 1.1498504958, 1.5090212516,$
$1.7795081538.$

*11h.*

$$P(x) = x^{11} + 4x^{10} - 4x^9 - 29x^8 - x^7 + 77x^6 + 18x^5 - 91x^4 - 17x^3 + 44x^2 + 2x - 5$$

with zeros

$-2.3251675821, -2.1608729421, -1.8987685612, -1.5128022260, -1.0689144474, -0.3943833319, 0.4081648226, 0.7726791475, 1.1332542970, 1.3771352470, 1.6696755768.$

*11i.*

$$P(x) = x^{11} + 5x^{10} - 33x^8 - 31x^7 + 72x^6 + 96x^5 - 53x^4 - 94x^3 + 2x^2 + 21x + 1$$

with zeros

$-2.3527441399, -2.2033238177, -1.8507454309, -1.5537240492, -1.2074756107, -0.7095504064, -0.0483319615, 0.5188178851, 1.2776078662, 1.4991492582, 1.6303204070.$

*11j.*

$$P(x) = x^{11} + 4x^{10} - 4x^9 - 28x^8 + x^7 + 71x^6 + 9x^5 - 76x^4 - 5x^3 + 29x^2 - 2x - 1$$

with zeros

$-2.4231269341, -2.2555569878, -1.9682631275, -1.3683048726, -0.9351128759, -0.1567877999, 0.2549659844, 0.7179913096, 1.1029061933, 1.4692549769, 1.5620341337.$

*10a.*

$$P(x) = x^{10} + x^9 - 10x^8 - 9x^7 + 35x^6 + 28x^5 - 49x^4 - 35x^3 + 21x^2 + 15x + 1$$

with zeros

$-2.0438978324, -1.8747379160, -1.6061643667, -0.8871053023, -0.5405487734, -0.0755542457, 0.94940842645, 1.3926174161, 1.7750769518, 1.9109056421.$

*10b.*

$$P(x) = x^{10} - 11x^8 + 43x^6 + x^5 - 70x^4 - 5x^3 + 41x^2 + 5x - 4$$

with zeros

$-2.0512046987, -1.8901235119, -1.4436115275, -0.9240095320, -0.4530750331, 0.2754227282, 1.0436366222, 1.6761643851, 1.8203440894, 1.9464564782.$

*10c.*

$$P(x) = x^{10} + x^9 - 10x^8 - 8x^7 + 36x^6 + 21x^5 - 55x^4 - 20x^3 + 31x^2 + 5x - 3$$

with zeros

$-2.0814722707, -1.9354760260, -1.5941925453, -1.0645615335, -0.4153057078,$
$0.2723856048, 0.9068222974, 1.3736058978, 1.6436726134, 1.8945216699.$

*10d.*

$$P(x) = x^{10} + x^9 - 10x^8 - 8x^7 + 36x^6 + 21x^5 - 56x^4 - 20x^3 + 36x^2 + 5x - 7$$

with zeros

$-2.1074832473, -1.9601233199, -1.4499029096, -1.1037091914, -0.5919971149,$
$0.5439985357, 0.8123137584, 1.2377915273, 1.7436913024, 1.8754206594.$

*10e.*

$$P(x) = x^{10} + 3x^9 - 7x^8 - 24x^7 + 16x^6 + 67x^5 - 12x^4 - 75x^3 + 27x + 1$$

with zeros

$-2.2804336186, -2.1741319877, -1.6456103621, -1.3463850232, -0.8922056157,$
$-0.0371787653, 0.7885925730, 1.2513716487, 1.6335012639, 1.7024798871.$

*10f.*

$$P(x) = x^{10} + 4x^9 - 2x^8 - 22x^7 - 7x^6 + 41x^5 + 19x^4 - 29x^3 - 11x^2 + 6x + 1$$

with zeros

$-2.2830670725, -2.0952939852, -1.7519056914, -1.2513509116, -0.7376403052,$
$-0.1440129992, 0.4772599965, 0.9163839954, 1.3556742939, 1.5139526794.$

*10g.*

$$P(x) = x^{10} + 3x^9 - 6x^8 - 21x^7 + 11x^6 + 49x^5 - 5x^4 - 42x^3 - 3x^2 + 10x + 2$$

with zeros

$-2.3063743564, -2.0918892509, -1.7169649617, -1.1098159312, -0.5077253146,$
$-0.2316352538, 0.6630389236, 1.0735272263, 1.5365135166, 1.6913254021.$

*10h.*

$$P(x) = x^{10} + 4x^9 - 3x^8 - 25x^7 - 4x^6 + 53x^5 + 16x^4 - 43x^3 - 8x^2 + 11x - 1$$

with zeros

$-2.3240858156, -2.2279781882, -1.9202396746, -1.2168095721, -0.8427208823,$
$0.1025614962, 0.4388667003, 0.9415494476, 1.4264161591, 1.6224403297.$

*10i.*

$$P(x) = x^{10} + 4x^9 - 3x^8 - 26x^7 - 7x^6 + 56x^5 + 28x^4 - 44x^3 - 19x^2 + 10x + 1$$

with zeros

$-2.3660875379, -2.0880295262, -1.7198029587, -1.4452313421, -0.8781886932,$
$-0.0883449095, 0.4266393485, 0.9652405430, 1.5621136422, 1.6316914341.$

*10j.*

$$P(x) = x^{10} + 4x^9 - 2x^8 - 22x^7 - 8x^6 + 39x^5 + 22x^4 - 23x^3 - 13x^2 + 2x + 1$$

with zeros

$-2.4197359388, -2.0445108209, -1.6602037609, -1.2149167136, -0.5754594269,$
$-0.2777262931, 0.3084897699, 0.9447993562, 1.3779225737, 1.5613412546.$

*10k.*

$$P(x) = x^{10} + 3x^9 - 6x^8 - 19x^7 + 15x^6 + 40x^5 - 20x^4 - 29x^3 + 11x^2 + 4x - 1$$

with zeros

$-2.4322742072, -2.2530467478, -1.6885320286, -1.0542727607, -0.3811054496,$
$0.2022810150, 0.5453485230, 1.1646939371, 1.3810819455, 1.5158257734.$

*10l.*

$$P(x) = x^{10} + 4x^9 - 3x^8 - 24x^7 - 2x^6 + 49x^5 + 11x^4 - 36x^3 - 8x^2 + 6x + 1$$

with zeros

$-2.4712101132, -2.3245762644, -1.8523125562, -1.0852867876, -0.5370156083,$
$-0.1573706315, 0.4543765122, 1.0754590433, 1.3974056424, 1.5005307633.$

*10m.*

$$P(x) = x^{10} + 5x^9 + x^8 - 26x^7 - 22x^6 + 46x^5 + 43x^4 - 31x^3 - 23x^2 + 6x + 1$$

with zeros

$-2.4769087010, -2.3479973966, -1.9465141781, -1.3310860646, -0.8789858773,$
$-0.1210141227, 0.3310860646, 0.9465141781, 1.3479973966, 1.4769087010.$

*10n.*

$$P(x) = x^{10} + 5x^9 + x^8 - 26x^7 - 21x^6 + 49x^5 + 40x^4 - 42x^3 - 20x^2 + 15x - 1$$

with zeros

$-2.4830549830, -2.2552293371, -1.9019166606, -1.4696796028, -1.0753382072,$
$0.0753382073, 0.4696796028, 0.9019166607, 1.2552293371, 1.4830549830.$

*10o.*

$$P(x) = x^{10} + 5x^9 - 30x^7 - 20x^6 + 66x^5 + 46x^4 - 63x^3 - 28x^2 + 21x + 1$$

with zeros

$-2.4865786888, -2.3779882051, -2.0652263242, -1.5582404660, -0.9548176386,$
$-0.0451823614, 0.5582404660, 1.0652263242, 1.3779882051, 1.4865786888.$

*10p.*

$$P(x) = x^{10} + 5x^9 + 2x^8 - 22x^7 - 22x^6 + 32x^5 + 35x^4 - 19x^3 - 16x^2 + 4x + 1$$

with zeros

$-2.4887280829, -2.1228013637, -1.8267242189, -1.3888803420, -0.8334230502,$
$-0.1665769498, 0.3888803420, 0.8267242189, 1.1228013637, 1.4887280829.$

*9a.*

$$P(x) = x^9 + 2x^8 - 7x^7 - 14x^6 + 15x^5 + 30x^4 - 10x^3 - 19x^2 + 2x + 1$$

with zeros

$-2.0637615881, -1.7878423787, -1.4737295601, -1.1681485032, -0.1950173935,$
$0.2865734724, 0.9501006842, 1.6083079843, 1.8435172829.$

*9b.*

$$P(x) = x^9 + x^8 - 9x^7 - 7x^6 + 28x^5 + 15x^4 - 34x^3 - 10x^2 + 12x + 1$$

with zeros

$-2.0828155591, -1.9132996185, -1.4849569391, -0.8392381638, -0.0795301693,$
$0.6715045970, 1.2671560185, 1.5910633246, 1.8701165096.$

*9c.*

$$P(x) = x^9 + x^8 - 9x^7 - 7x^6 + 28x^5 + 15x^4 - 34x^3 - 9x^2 + 12x - 1$$

with zeros

$-2.1309169868, -1.8229200392, -1.5033676060, -0.9250602306, 0.0917272020,$
$0.5159964779, 1.3003471006, 1.6377903211, 1.8364037610.$

*9d.*

$$P(x) = x^9 + 3x^8 - 5x^7 - 20x^6 + 3x^5 + 41x^4 + 12x^3 - 25x^2 - 10x + 1$$

with zeros

$-2.1644737293, -1.8442489311, -1.5490152781, -1.2779544232, -0.5617697916,$
$0.0834767362, 0.9695427968, 1.5582730860, 1.7861695343.$

*9e.*

$$P(x) = x^9 + 2x^8 - 8x^7 - 14x^6 + 23x^5 + 30x^4 - 28x^3 - 18x^2 + 12x - 1$$

with zeros

$-2.1756386468, -2.0829814779, -1.6589620064, -1.0511454981, 0.1006233920, 0.4097639658, 1.0747533499, 1.6103493760, 1.7732375456.$

*9f.*

$$P(x) = x^9 + 2x^8 - 7x^7 - 13x^6 + 16x^5 + 26x^4 - 13x^3 - 17x^2 + 3x + 3$$

with zeros

$-2.1902962455, -1.9543227711, -1.3754424820, -0.7897044126, -0.4668481743, 0.5609064534, 0.9056522861, 1.5194783193, 1.7905770267.$

*9g.*

$$P(x) = x^9 + 2x^8 - 8x^7 - 15x^6 + 22x^5 + 37x^4 - 23x^3 - 32x^2 + 6x + 5$$

with zeros

$-2.1960921093, -1.9275670118, -1.5685583592, -1.1022910275, -0.3749141298, 0.4924200452, 1.3091165857, 1.5888335723, 1.7790524345.$

*9h.*

$$P(x) = x^9 + 2x^8 - 7x^7 - 13x^6 + 16x^5 + 26x^4 - 13x^3 - 16x^2 + 3x + 2$$

with zeros

$-2.2381922257, -1.8263788987, -1.4815453746, -0.8498548995, -0.3251309941, 0.4690037900, 0.9220119683, 1.5726140274, 1.7574726069.$

*9i.*

$$P(x) = x^9 + 2x^8 - 8x^7 - 13x^6 + 25x^5 + 26x^4 - 35x^3 - 14x^2 + 16x - 1$$

with zeros

$-2.2511086184, -2.0681995630, -1.7466922116, -0.9604695896, 0.0670596447, 0.6662263119, 1.2088923611, 1.4543327695, 1.6299588953.$

*9j.*

$$P(x) = x^9 + 2x^8 - 7x^7 - 12x^6 + 17x^5 + 22x^4 - 15x^3 - 13x^2 + 3x + 1$$

with zeros

$-2.3107856863, -2.0218147634, -1.2621399491, -0.8119166225, -0.2049697063, 0.3786661691, 1.1066160372, 1.4552047072, 1.6711398142.$

*9k.*

$$P(x) = x^9 + 3x^8 - 5x^7 - 18x^6 + 7x^5 + 34x^4 - x^3 - 20x^2 - 3x + 1$$

with zeros

$-2.3211043625, -2.0571562350, -1.6357383948, -0.8831880500, -0.3649696986, 0.1629715947, 1.0697851735, 1.3880380382, 1.6413619345.$

*9l.*

$$P(x) = x^9 + 3x^8 - 6x^7 - 20x^6 + 13x^5 + 44x^4 - 13x^3 - 34x^2 + 4x + 7$$

with zeros

$-2.3409054170, -2.1355500720, -1.8654464935, -0.9276666942, -0.5165162725, 0.6016630323, 1.1021179319, 1.4284992653, 1.6538047198.$

*9m.*

$$P(x) = x^9 + 3x^8 - 5x^7 - 17x^6 + 9x^5 + 30x^4 - 8x^3 - 16x^2 + 3x + 1$$

with zeros

$-2.3426438056, -2.2074781887, -1.5694612587, -0.9169684076, -0.1823771424, 0.3853539490, 0.8805752874, 1.3823053107, 1.5706942559.$

*9n.*

$$P(x) = x^9 + 4x^8 - 2x^7 - 22x^6 - 8x^5 + 38x^4 + 20x^3 - 21x^2 - 9x + 1$$

with zeros

$-2.3461032020, -2.1157828000, -1.6655379636, -1.3188507934, -0.5065796134, 0.0930188077, 0.8931217035, 1.3380272439, 1.6286866176.$

*9o.*

$$P(x) = x^9 + 4x^8 - 2x^7 - 21x^6 - 5x^5 + 36x^4 + 10x^3 - 21x^2 - x + 1$$

with zeros

$-2.4371859729, -2.1013665640, -1.7246624253, -1.3584401616, -0.2413218249, 0.2131112495, 0.8115176266, 1.2883314491, 1.5500166236.$

*9p.*

$$P(x) = x^9 + 5x^8 + 2x^7 - 22x^6 - 23x^5 + 28x^4 + 34x^3 - 10x^2 - 11x + 1$$

with zeros

$-2.4379841177, -2.2714065535, -1.8656159312, -1.2268101121, -0.7284138103, 0.0862588381, 0.6551291802, 1.2281126797, 1.5607298269.$

*9q.*

$$P(x) = x^9 + 4x^8 - 2x^7 - 21x^6 - 5x^5 + 37x^4 + 12x^3 - 24x^2 - 5x + 4$$

with zeros

$$-2.4551506406, -2.1489402020, -1.6832772711, -1.1508618142, -0.6086635831, 0.3766154375, 0.8873964847, 1.2810143088, 1.5018672800.$$

*9r.*

$$P(x) = x^9 + 3x^8 - 5x^7 - 17x^6 + 9x^5 + 31x^4 - 7x^3 - 18x^2 + x + 1$$

with zeros

$$-2.4647771822, -2.0331742862, -1.5949221441, -0.9086789963, -0.2273072183, 0.2687899541, 1.1308782804, 1.3233906191, 1.5058009735.$$

*9s.*

$$P(x) = x^9 + 5x^8 + 2x^7 - 22x^6 - 23x^5 + 28x^4 + 34x^3 - 9x^2 - 9x + 1$$

with zeros

$$-2.4744054675, -2.2217160311, -1.8303551495, -1.3530410647, -0.6239901009, 0.1048150647, 0.5596488801, 1.3280517078, 1.5109921613.$$

## References

1. R. M. Robinson, *Algebraic equations with span less than* 4, Math. of Comp. **18** (1964), 547–559.