POLYNOMIAL INEQUALITIES

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Introduction.

Between 1974 and 1989 I wrote some papers where I proved a number of inequalities, some of them conjectured by others (Erdős, Hayman, H. Kuhn). All inequalities are sharp, i.e. they may be obtained by finding an extreme value of a functional. The (algebraic or trigonometrical) polynomials may satisfy various restrictions with respect to (some) norm, location of zeros etc.

I used a variational method to find these extreme values, i.e. I first proved that there existed at least one polynomial for which the functional assumed the extreme value, next I characterized this "extremizing" polynomial by using the fact that a small change of the polynomial (respecting the restrictions) could not improve the polynomial.

Such methods have been used extensively also by other authors (Vladimir Markov and Erdős, for instance), but not in a systematic way. I tried to construct a generally applicable method, and the result of this effort can be found in my dissertation [1] (I would here like to mention the secretarial help in TEXnifying my manuscript arranged by Professor Christian Berg).

Unfortunately, proofs using this variational method tend to be complicated, mostly because the possibility of multiple zeros of the extremizing polynomial has to be investigated. Further complications arise when the functional depends also on higher derivatives of the polynomial.

Proof of a generalization of a conjecture of Erdős.

Already the first paper in the series ([2]) shows these difficulties. I was invited by Professor Rahman to give some lectures on the paper at Université de Montréal, so I am confident that the proofs in the paper are correct. I had very valuable discussions with Professor Rahman who inspired much of my work in the following months.

A few more comments on [2].

The functional to be maximized is

$$f(T) = \max_{a,b} \left(\int_a^b dx\, |T(x)| \Big/ (b-a) \right) \Big/ (\max_{|x|\le\pi} |T(x)|), \tag{1}$$

where a and b are two consecutive zeros of the trigonometrical polynomial T. The restrictions on T are that T should be a trigonometric polynomial of degree $n \ge 1$ with $2n$ real zeros in the interval $(-\pi, \pi]$.

To easily prove the existence of a maximizing polynomial one reformulates the problem:

We impose the further restriction that

$$\max_{|x|\leq\pi} |T(x)| = 1 \tag{2}$$

and rewrite the functional as

$$f(T) = \max_{a,b} \int_0^1 dt\, |T(a + t(b - a))|. \tag{3}$$

Clearly, f is a continuous functional defined on a compact set and therefore assumes its maximal value.

Usually the proof of existence of the extremizing function is not so easy, but it should always be part of the investigation.

The parameters of the problem are the zeros of T and the choice of the root interval (a, b). However, during the variational process the natural way to characterize the maximizing polynomial T_m is by means of its stationary values. In fact, these turn out to be the simplest possible, namely $\{-1, 1\}$, which shows that $T_m(x) = A \sin(n(x - c))$ for some real numbers A and c.

Further applications of the variational method.

In my next paper ([3]) I developed the variational method further and proved a number of conjectures, often in a generalized form. Unfortunately, mostly due to my own inexperience, quite a number of errors crept in when I stated or proved theorems. The referee taught me a lot about the British mathematical vocabulary, but did not comment on my mathematics, and I was simply too lazy to check it if others did not consider it necessary.

I must of course assume full responsibility for the errors, but at least one reader felt that he should complain directly to the journal. This resulted in a very interesting and completely elementary completion (by Sheil-Small) of the proof of Lemma 2 of [3]. This is described in [4].

During proof reading I found more (also rather trivial) errors. These were corrected in a continuation ([5]) of [3]. Also a new proof (the old one was almost unreadable) of the algebraic part of Theorem 5 of [3] was given. Incidentally, most of the old results were generalized.

There are a few more papers on polynomials, but all results could be considered as special cases of theorems valid for entire functions, so that it seems more natural to discuss these matters in the section on these functions.

References

[1] G. K. Kristiansen, *A method for solution of extremal problems for entire functions, in particular polynomials*, Thesis.
[2] ———, *Proof of an inequality for trigonometric polynomials*, Proc. Amer. Math. Soc. **44** (1974), 49–57.
[3] ———, *Some inequalities for algebraic and trigonometric polynomials*, J. London Math. Soc. (2) **20** (1979), 300–314.

[4] ———, *Corrigendum: Some inequalities for algebraic and trigonometric polynomials*, J. London Math. Soc. (2) **22** (1980), 575–576.

[5] ———, *Some inequalities for algebraic and trigonometric polynomials II*, J. London Math. Soc. (2) **28** (1983), 83–92.