VARIOUS CONCERNING THE DISTRIBUTION MODULO ONE OF GEOMETRIC SEQUENCES

Gundorph K. Kristiansen

Abstract. A number of examples of non-uniform distribution modulo one of geometric sequences are given.

Introduction

For all real numbers x we define the fractional part $\mathrm{fr}(x)$ as $x - [x]$. For various real numbers $\alpha \in (1, \infty)$ we study the dependence of the sequence $(\mathrm{fr}(\lambda\alpha^n) \mid n = 0, 1, \dots)$ on λ. We are particularly interested in examples of non-uniform distribution of the members of such a sequemce.

For any $\alpha > 1$ we can obtain a non-uniform distribution

We choose a natural number m such that $\sqrt[m]{m+1} < \alpha$.

Next we define the sequence of natural numbers $(g_p \mid p = 0, 1, \dots)$ recursively:

The first number, g_0, is arbitrary.

Then, $g_{p+1} = 1 + [\alpha^m g_p]$.

We define, for $p = 0, 1, \dots$, the closed interval $I_p = [g_p/\alpha^{mp}, (g_p + 1/(\alpha^m - 1))/\alpha^{mp}]$.

Evidently, for all p, $I_{p+1} \subset I_p$.

We put $\{\lambda\} = \cap_{p=0}^{\infty} I_p$.

Then, for all p, $g_p \leq \lambda\alpha^{mp} \leq g_p + 1/(\alpha^m - 1)$, i.e. $\mathrm{fr}(\lambda\alpha^{mp}) \leq 1/(\alpha^m - 1)$.

Thus, for $N \to \infty$, the fraction of the numbers $\lambda\alpha^n$ $(n = 0, 1, \dots, N)$ having fractional parts not greater than $1/(\alpha^m - 1)$ is not less than $1/m$, a fact incompatible with a uniform distribution modulo one, since $1/(\alpha^m - 1) < 1/m$.

Distributions contained in subintervals

A special kind of non-uniform distribution modulo one is a distribution contained in a subinterval, typically $[0, \frac{1}{2})$ is chosen.

I read Mahler's interesting paper [1], treating the particular case $\alpha = 3/2$. He shows that there can be at most a countable number of values of λ, for which this is possible, and he also finds a number of rather severe conditions such a λ must satisfy. No solution is found, however, and it appears improbable, on the basis of the results of extensive numerical experiments, that there is one.

Mahler suggests that one should consider other values of α, for instance rational.

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Obviously, the problem is to construct a sequence $(g_n \mid n = 0, 1, \dots)$ of natural numbers such that the intersection $\cap_{n=0}^{\infty}[g_n/(\alpha^n), (g_n + \frac{1}{2})/(\alpha^n))$ is non-empty.

For $\alpha \geq 3$ we can obtain a nested sequence of intervals simply by choosing g_0 arbitrary and $g_{n+1} = -[-\alpha g_n]$ for $n = 0, 1, \dots$, and so there is a λ with the property looked for.

To carry out this argument we use that any closed real interval of length one contains an integer. We can refine the argument if the endpoints of the interval are rational numbers, which we can assume have a common denominator $q > 1$. Then the interval need only have the length $1 - 1/q$ to with certainty contain an integer. We can then show that also to values of α of the form $3 - 2/m$, where $m = 3, 4, \dots$, there exist values of λ for which $0 \leq \mathrm{fr}(\lambda\alpha^n) < \frac{1}{2}$ for $n = 0, 1, \dots$.

The details: The integer g_{n+1} should belong to the closed interval $[\alpha g_n, \alpha g_n + (\alpha - 1)\frac{1}{2}]$, where $(\alpha - 1)\frac{1}{2} = 1 - 1/m$. The argument works also for $m = 2$, but gives the trivial result that λ is an integer.

It is easy to give examples of numbers α, for which solution is impossible. Take, for instance, the number $(1 + \sqrt{5})/2$. It is a root of the equation $\alpha^2 = \alpha + 1$. The fractional parts $r_n = \mathrm{fr}(\lambda\alpha^n)$ must satisfy the recurrence $r_{n+2} = r_{n+1} + r_n$, since the right hand side is a number in $[0, 1)$ and so does not differ from a number in $[0, \frac{1}{2})$ by a non-zero integer. But then r_n is a linear combination of the two numbers $((1 \pm \sqrt{5})/2)^n$. However, α^n cannot occur with a non-zero coefficient, since r_n is bounded, and the sign of $((1 - \sqrt{5})/2)^n$ alternates, so that r_n cannot belong to $[0, \frac{1}{2})$ for all n.

I communicated these results (and a number of others) to Kurt Mahler in 1969.

I think that the most interesting case is the one where α is the square root of one of the numbers $3, 5, 6, 7, 8$.

In general, if α is the square root of a positive integer $m > 2$, it appears natural to use the expansions of both λ and $\lambda\sqrt{m}$ in powers of m^{-1},

$$\lambda = g_0 + \sum_{j=1}^{\infty} g_j \, m^{-j}, \tag{1}$$

and

$$\lambda\sqrt{m} = h_0 + \sum_{k=1}^{\infty} h_k \, m^{-k}. \tag{2}$$

Here g_0 and h_0 are arbitrary integers, while the other coefficients in general should be integers belonging to the interval $[0, m-1]$. However, since both $\mathrm{fr}(\lambda m^n)$ and $\mathrm{fr}(\lambda\sqrt{m}\, m^n)$ must belong to the interval $[0, \frac{1}{2})$ for $n = 0, 1, \dots$, these coefficients are restricted to the interval $[0, [(m - 1)/2]]$.

We see that there is an important difference between even and odd values of m: For m even the highest value of a coefficient is $m/2 - 1$, so that actually the mentioned fractional parts are at most equal to $\frac{1}{2}(m - 2)/(m - 1)$. For m odd, the highest permitted value of a coefficient is $(m - 1)/2$, and we do not allow a representation with all coefficients of sufficiently high index assuming this maximal value. Unfortunately, I have not been able to rule out that possibility, so that, in fact, I have only obtained $0 < \mathrm{fr}(\lambda\alpha^n) \leq \frac{1}{2}$ for $n = 0, 1, \dots$, when $\alpha = \sqrt{m}$ and $m \in \{3, 5, 7\}$.

We must ensure that the two expansions (1) and (2) are compatible, i.e. we must have

$$h_0 + \sum_{k=1}^{\infty} h_k\, m^{-k} = \left(g_0 + \sum_{j=1}^{\infty} g_j\, m^{-j} \right) \sqrt{m}. \tag{3}$$

This makes us consider the general situation where two expansions like the right hand sides of (1) and (2) are given, and we wish to combine them to obtain an arbitrary real number. For instance, if m is odd, an arbitrary number can be obtained by addition. Actually, an integral coefficient in the interval $[0, m-1]$ can be obtained, usually in several ways, as the sum of two integers in the interval $[0, (m-1)/2]$ (the corresponding statement is not true for m even).

For m odd, I shall show that an arbitrary real number (and thus $\sqrt{m}$) can also be obtained as the ratio between two expansions of the type occurring in the right hand sides of (1) and (2). It suffices to show that for an arbitrary real number $A \in (1, m)$ it is possible to find coefficients such that

$$h_0 + \sum_{k=1}^{\infty} h_k\, m^{-k} = \left(g_0 + \sum_{j=1}^{\infty} g_j\, m^{-j} \right) A. \tag{4}$$

To do this, we start by finding integers g_0, g_1, and h_0 such that

$$-\frac{1}{2} < h_0 - (g_0 + g_1/m)A < A/(2m). \tag{5}$$

First, g_0 is chosen as an arbitrary positive integer. Next, h_0 is defined (uniquely) as an integer such that

$$-\frac{1}{2} < h_0 - g_0\, A \leq \frac{1}{2}.$$

Then g_1 is chosen as the smallest integer in $[0, (m-1)/2]$ such that

$$h_0 - (g_0 + g_1/m)A < A/(2m).$$

If this were not possible, we would have

$$\frac{m-1}{2} A/m \leq \frac{1}{2} - A/(2m),$$

i.e. $A \leq 1$, which is not true.

We must still check if this choice of g_1 would make

$$h_0 - (g_0 + g_1/m)A \leq -\frac{1}{2}.$$

If this were the case, the minimality of g_1 would imply that

$$A/m \geq \frac{1}{2} + A/(2m),$$

i.e. $A \geq m$, which is not true either.

This was step 0 of the solution of (4).

To describe step n, we first introduce an abbreviation: Define

$$S_n = h_0 + \sum_{k=1}^{n} h_k\, m^{-k} - \left(g_0 + \sum_{j=1}^{n+1} g_j\, m^{-j} \right) A. \tag{6}$$

In step n we assume that we have obtained

$$-\frac{1}{2}\, m^{-n+1} < S_{n-1} < \frac{A}{2m^n}. \tag{7}$$

We shall then choose the integers h_n and g_{n+1} in $[0, (m-1)/2]$ such that we get the inequalities (7) with $n-1$ replaced by n, so that the argument can continue.

Thus $S_{n-1} m^{n-1}$ belongs to the interval $(-\frac{1}{2}, A/(2m))$, and we shall show that for each point t in this interval it is possible to find h_n and g_{n+1} such that $t + h_n/m - g_{n+1}\, A/m^2$ belongs to the interval $(-1/(2m), A/(2m^2))$. But this could also be expressed in another way: We form the union of the tranlations of the interval $(-1/(2m), A/(2m^2))$ by all numbers $-h_n/m + g_{n+1}\, A/m^2$ and show that this union contains the interval $(-\frac{1}{2}, A/(2m))$.

Let us first keep h_n fixed and consider the union

$$\bigcup_{g_{n+1}=0}^{(m-1)/2} \left(\frac{1}{m} \left(-h_n - \frac{1}{2} + \frac{g_{n+1}}{m}\, A \right), \frac{1}{m} \left(-h_n + \frac{A}{m} \left(\frac{1}{2} + g_{n+1} \right) \right) \right). \tag{8}$$

The intervals overlap, since $A < m$, and so the union in (8) equals

$$\left(\frac{1}{m} \left(-h_n - \frac{1}{2} \right), \frac{1}{m} \left(-h_n + \frac{A}{2} \right) \right). \tag{9}$$

Finally we form

$$\bigcup_{h_n=0}^{(m-1)/2} \left(\frac{1}{m} \left(-h_n - \frac{1}{2} \right), \frac{1}{m} \left(-h_n + \frac{A}{2} \right) \right). \tag{10}$$

Again the intervals overlap, this time because $A > 1$, and the union in (10) equals $(-\frac{1}{2}, \frac{A}{2m})$, as expected, and the proof by induction is concluded.

In the case where m is even, A must be restricted somewhat. We require

$$\frac{m}{m-2} < A < m - 2. \tag{11}$$

Now we can use a slightly modified method to prove that a solution exists (note that $m > 4$, so that (11) has meaning).

We showed earlier that the fractional parts we are interested in, now belong to the interval $[0, \sigma]$, where we have defined

$$\sigma = \frac{m-2}{2m-2}, \tag{12}$$

instead of the interval $[0, \frac{1}{2})$.

The changes to the arguments used in the case where m was odd, largely consist in replacing $\frac{1}{2}$ by σ. But I shall give the details:

(5) is replaced by

$$-\sigma < h_0 - (g_0 + g_1/m)A < \sigma\, A/m. \tag{5'}$$

We start by having g_0 and h_0 satisfy

$$-\sigma < h_0 - g_0\, A \leq 1 - \sigma.$$

Next g_1 is chosen as the smallest integer in $[0, \frac{m}{2} - 1]$ for which

$$h_0 - (g_0 + g_1/m)A < \sigma\, A/m.$$

If this were not possible, we would have

$$\left(\frac{m}{2} - 1\right)\frac{A}{m} \leq 1 - \sigma - \sigma\,\frac{A}{m},$$

which can be simplified to $\frac{A}{m} \leq \frac{1}{m-2}$, contradicting (11).

If this choice of g_1 makes

$$h_0 - (g_0 + g_1/m)A \leq -\sigma,$$

we must have

$$\frac{A}{m} \geq \sigma + \sigma\frac{A}{m},$$

implying $A \geq m - 2$, again contradicting (11).

In step n there are the following changes:

The inequalities (7) become

$$-\sigma\, m^{-n+1} < S_{n-1} < \sigma\frac{A}{m^n}. \tag{7'}$$

(8) is changed to

$$\bigcup_{g_{n+1}=0}^{\frac{m}{2}-1}\left(\frac{1}{m}\left(-h_n - \sigma + \frac{g_{n+1}}{m}\,A\right), \frac{1}{m}\left(-h_n + \frac{A}{m}\,(\sigma + g_{n+1})\right)\right). \tag{8'}$$

Overlap requires $\frac{A}{m} - \sigma < \frac{A}{m}\sigma$, i.e. $A < m - 2$, satisfied according to (11).

(9) becomes

$$\left(\frac{1}{m}\,(-h_n - \sigma)\,, \frac{1}{m}\,(-h_n + A\sigma)\right), \tag{9'}$$

so that (10) is replaced by

$$\bigcup_{h_n=0}^{\frac{m}{2}-1}\left(\frac{1}{m}\,(-h_n - \sigma)\,, \frac{1}{m}\,(-h_n + A\sigma)\right). \tag{10'}$$

Overlap requires $1-\sigma < A\sigma$, i.e. $A > \frac{m}{m-2}$, satisfied according to (11). Thus (10') equals $(-\sigma, \sigma\frac{A}{m})$, as it should.

We finally note that since $m > 4$, the number $A = \sqrt{m}$ satisfies (11), which concludes the proof.

If $\alpha = \sqrt{2}$, obviously we cannot have $0 \le \mathrm{fr}(\lambda\alpha^n) < \frac{1}{2}$ for $n = 0, 1, \ldots$ for any real λ.

We can, however, obtain $0 \le \mathrm{fr}(\lambda\alpha^n) < \frac{2}{3}$ for $n = 0, 1, \ldots$.

To prove this, we use a method somewhat resembling the one used above for the other square roots. The problem is here to show that any real number in $(1, 2)$ (and so also $\sqrt{2}$) can be represented as the ratio between two expansions

$$h_0 + \sum_{k\in S} 2^{-k} \tag{13}$$

and

$$g_0 + \sum_{j\in T} 2^{-j}, \tag{14}$$

where S and T are subsets of $\mathbb{N}$, while h_0 and g_0 are positive integers. We further require that neither S nor T contains consecutive integers, and that if one of them contains a number N, it does not also contain all numbers greater than N and having the same parity. Then we are sure that, for instance,

$$0 \le \mathrm{fr}\left(2^n \sum_{k\in S} 2^{-k}\right) < \frac{2}{3}$$

for $n = 0, 1, \ldots$.

For any number $A \in (1, 2)$ we must solve an equation, similar to (4),

$$h_0 + \sum_{k\in S} 2^{-k} = A\left(g_0 + \sum_{j\in T} 2^{-j}\right). \tag{15}$$

As above, we solve for the coefficients recursively.

In step n we assume given that we have found finite subsets $S_{n-1} \subset S$ and $T_{n-1} \subset T$ with k_{n-1} the maximal member of S_{n-1} and j_{n-1} the maximal member of T_{n-1}, such that

$$0 < h_0 + \sum_{k\in S_{n-1}} 2^{-k} - A\left(g_0 + \sum_{j\in T_{n-1}} 2^{-j}\right) < \frac{1}{3}2^{-k_{n-1}}. \tag{16}$$

(I shall disregard the (real) possibility that the leftmost inequality becomes an equality. In fact, the problem would then be solved with A a rational number). In (16) it is understood that S_0 and T_0 are empty, while k_0 and j_0 equal 0. Thus, for $n = 1$ we are supposed to have found the integers g_0 and h_0 such that $0 < h_0 - Ag_0 < \frac{1}{3}$, which is always possible. For instance we could, for $A = \sqrt{2}$, choose $h_0 = 3$ and $g_0 = 2$.

In step n we now determine the set $T_n \setminus T_{n-1}$. We shall do this such that the expression

$$h_0 + \sum_{k \in S_{n-1}} 2^{-k} - A\left(g_0 + \sum_{j \in T_n} 2^{-j}\right) \tag{17}$$

becomes negative but with minimal numerical value. But then $A \sum_{j \in T_n \setminus T_{n-1}} 2^{-j}$ need not be greater than $\frac{1}{3}2^{-k_{n-1}}$. And since $A > 1$, no member of $T_n \setminus T_{n-1}$ is smaller than $k_{n-1} + 2$. In particular, $j_n \geq k_{n-1} + 2$. We also see that the value of the expression (17) cannot be less than $-\frac{A}{3}2^{-j_n}$ since, otherwise, its numerical value could be decreased, if we removed j_n from T_n and instead inserted some finite subset of $\{j_n + 1, j_n + 3, \dots\}$.

Next we determine the set $S_n \setminus S_{n-1}$ such that

$$h_0 + \sum_{k \in S_n} 2^{-k} - A\left(g_0 + \sum_{j \in T_n} 2^{-j}\right)$$

becomes minimally positive. But then $\sum_{k \in S_n \setminus S_{n-1}} 2^{-k}$ need not be greater than $\frac{A}{3}2^{-j_n}$. Since $A < 2$, this means that no member of $S_n \setminus S_{n-1}$ is smaller than $j_n + 1$, and so not smaller than $k_{n-1} + 3$, so that T_n is permissible. We also see that no member of $T_{n+1} \setminus T_n$ is smaller than $k_n + 2$, and so not smaller than $j_n + 3$, which shows that also T_{n+1} is permissible. So is, of course, T_1. This concludes the proof.

One can prove that λ can be determined for some algebraic numbers satisfying various conditions. Suppose, for instance, that α is a Pisot number with the positive number β as its numerically largest conjugate. Then we can choose $\lambda = (\alpha - 1)\alpha^m$ with m a sufficiently large positive number and get
$\mathrm{fr}(\lambda\alpha^n) = (1 - \beta)\beta^{m+n} + \sum_j (1 - \beta_j)\beta_j^{m+n}$, where the sum gives the contribution from the remaining conjugates, which becomes arbitrarily small compared to the contribution from β when $m + n$ increases, and so we can obtain $\mathrm{fr}(\lambda\alpha^n) \in (0, \frac{1}{2})$ for all integers $n \geq 0$.

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References

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