# UNIT DISTANCE GRAPHS IN A QUADRANGULAR GEOMETRY

Gundorph K. Kristiansen

October 10th 2003

Abstract. I discuss a norm-derived metric of the plane, corresponding to the unit circle (i.e. the set of points with distance 1 from the origin) being a parallelogram. In this case it is trivially possible to four-colour the plane in such a way that two points with distance 1 always receive different colours. However, there are an infinite number of essentially different colourations, and a classification of these is given.

## Unit distance graphs

A unit distance graph $G$ relative to a given metric $d$ of the plane is a collection $V$ of points in the plane together with a collection $E$ of 2-subsets of $V$ satisfying

$$\{P, Q\} \in E \implies d(P, Q) = 1.$$

An edge in $G$ is a straight line connecting two points $P$ and $Q$ where $\{P, Q\} \in E$.

## Norm-based metrics

It is well known (see for instance [2] and its references) that in such a metric the chromatic number of a unit distance graph $G$ in the plane is finite (less than 8), and so, according to a theorem by Erdős and de Bruijn (see [1]), there is a finite subgraph of $G$ with the same chromatic number.

So let us assume that the finite unit distance graph $H$ has maximal chromatic number. Obviously, $H$ is also unit distance graph in an infinite number of other norms. The simplest of these norms is obtained by only retaining the minimal number of points $P$ on the original unit circle, namely those for which the line $OP$ connecting the point to the origin is a translate of an edge in $H$. We form a new unit circle as the boundary of the convex hull of the set of these lines $OP$. This unit circle is symmetric with respect to the origin and defines a norm in which $H$ is a unit distance graph.

We conclude that to determine the maximal chromatic number for unit distance graphs relative to normbased metrics in the plane we need only investigate those norms in which the unit circle is a polygon.

1991 *Mathematics Subject Classification.* 05C15.

*Key words and phrases.* Unit distance graphs, Normed spaces.

## General considerations

In the following we shall not talk much about graphs, but concentrate on permissible colourations of the whole plane, i.e. colourations where two points with distance 1 have different colours. The plane is usually described as $\mathbb{R}^2$, but occasionally also as $\mathbb{C}$.

The norm function applies to vectors, but for simplicity we shall also consider it a point function, so that for an arbitrary point $P = (x, y) = z$ we define $\|P\| = \|(x, y)\| = \|z\| = \|OP\|$, i.e. as the distance from origo to the point.

Assume that the unit circle is a parallelogram. We can always find a linear transformation mapping this unit circle on a square which again is the unit circle for the supremum norm:

$$\|(x, y)\| = \max\{|x|, |y|\}. \tag{1}$$

This linear transformation, when applied to the whole plane, is an isometry of the plane with the original norm on the plane with the supremum norm, and obviously we can restrict attention to the latter when discussing colouration of unit distance graphs.

Also, in the following, we shall call two colourings $f$ and $g$ (understood as mappings of the plane into a fixed set of colours) equivalent if there is an isometry $i$ of the plane and a permutation $p$ of the set of colours such that $f = p \circ g \circ i$.

Only equivalence classes of colourings will be discussed.

## All four-colourings in the supremum norm

Divide the plane into classes

$$K_{a,b} = \{(a + m, b + n) \mid m, n \in \mathbb{Z}\} \quad (0 \le a, b < 1). \tag{2}$$

In each such class $K_{a,b}$ each of the points $(a, b), (a+1, b), (a, b+1), (a+1, b+1)$ has distance 1 from each of the others, and so four colours are necessary for each class.

In the remainder of this section the term "line"means a vertical or a horizontal line.

Assume that we have a permissible four-colouring of the plane.

Consider first

Case 1. No line in any $K_{a,b}$ contains three consecutive points with three different colours.

Thus any line in a given class is two-coloured. We shall show that this implies that any line in the plane is two-coloured.

In fact, we compare the two classes $K_{a,b}$ and $K_{a,d}$ with $0 \le b < d < 1$. Here the point $(a, b) \in K_{a,b}$ has distance one from the two points $(a+1, d)$ and $(a+1, d-1)$, both belonging to $K_{a,d}$, and so its colour must belong to the complement to the colourset of the line $x = a + 1$ in $K_{a,d}$. Thus the line $x = a$ in $K_{a,b}$ has the same colourset as the line $x = a$ in $K_{a,d}$, q.e.d.

For arbitrary real $t$ we define $S_t$ as the set of colours used to colour the points on the line $x = t$. We saw that $S_a$ was the complement of $S_{a+1}$ and so equal to $S_{a+2}$. The argument is applicable for all real $a$, and so $S_t$ is always a 2-set depending only on $t$ modulo 2.

Similarly the set of colours used for points on the line $y = u$ is a 2-set $T_u$ depending on $u$ modulo 2.

Clearly the point $(a, b)$ has a colour in $S_a \cap T_b$, which then cannot be empty. But the set $S_a \cap T_{b+1}$, which contains the colour of $(a, b+1)$, cannot be empty either, and so the colour of $(a, b)$ is simply the unique element in $S_a \cap T_b$.

Let the set of colours be $M = \{1, 2, 3, 4\}$.

We have

**Theorem 1.** *Each equivalence class of permissible four-colourings of the plane with the supremum norm and with each horizontal or vertical line two-coloured can be represented by a colouration determined in the following way:*

*To each $a \in [0, 1)$ we choose a colourset $S_a$ from the collection $S = \{\{1, 2\}, \{3, 4\}\}$, and to each $b \in [0, 1)$ we choose a colourset $T_b$ from the collection $T = \{\{1, 3\}, \{2, 4\}, \{1, 4\}, \{2, 3\}\}$.*

*For each real number $t$ we define the colourset*

$$S_t = \begin{cases} S_{t-[t]} & \text{for } [t] \text{ even} \\ \complement S_{t-[t]} & \text{for } [t] \text{ odd} \end{cases}$$

*with the complement taken relative to $M$.*

*For each real number $u$ we define the colourset*

$$T_u = \begin{cases} T_{u-[u]} & \text{for } [u] \text{ even} \\ \complement T_{u-[u]} & \text{for } [u] \text{ odd.} \end{cases}$$

*The colour of the point $(t, u)$ is then the unique member of $S_t \cap T_u$*

*Proof.* If two points $(t_1, u_1)$ and $(t_2, u_2)$ have distance 1, either $|t_1 - t_2|$ or $|u_1 - u_2|$ is equal to 1, and so either $S_{t_1} \cap S_{t_2}$ or $T_{u_1} \cap T_{u_2}$ is empty, and the colours of the two points must be different.

It remains to show that all possibilities are accounted for.

Clearly, both $S$ and $T$ contains with each member also its complement, and so each collection has an even number of members. Since there are 6 2-sets in all, one of the collections has 2 members. Choose the isometry $i$ of the definition of equivalence of colourings such that $S$ has two members, and choose the permutation $p$ such that these members are those of the theorem. The result follows.

A subcase is obtained, if one replaces $T$ by the subset $\{\{1, 3\}, \{2, 4\}\}$.

<u>Case 2</u>. There is a class $K_{a,b}$ with a horizontal or vertical line containing points with at least three different colours.

Without loss of generality we can assume that the colourset $M$ is $\{1, 2, 3, 4\}$, and that the colour function $C$ takes the values 1,2,3 in the points $(a-1, b)$, $(a, b)$, and $(a+1, b)$, respectively, where $a$ and $b$ are some real numbers (see Figure 1).

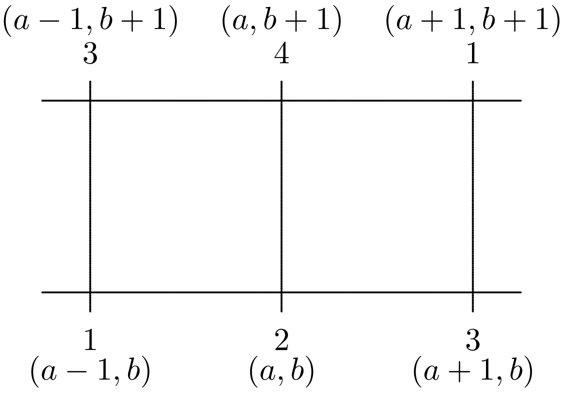


Figure 1

Clearly, $C(a,b+1)=4$, and so $C(a-1,b+1)=3$ and $C(a+1,b+1)=1$. The corresponding three values of $C$ on the line $y=b+2$ are the same as those on the line $y=b$ in the same order. Using the same argument as in Case 1, we see that the lines $x=a-1$, $x=a$, and $x=a+1$ are two-coloured. But then any line $x=a+n$, where $n$ is integral, is two-coloured.

And we can say more. We use the simple fact that on no line can we have three different colours for points in a closed interval of length 1. Thus, on the line $y=b$, if we define the real number $d\in[0,1]$ as the supremum of the numbers $d'$ such that all points in the interval $a-d'\le x\le a$ have colour 2, there cannot be a point $(x,b)$ with colour 3 for $a<x<a+1-d$ (otherwise there should exist two points on the line $y=b+1$ with distance 1 and both having the colour 4). We conclude that all points $(x,b)$ with $x\in(a-d,a+1-d)$ have colour 2. And exactly one of the endpoints must also have this colour.

If $a+1-d<x\le a+1$, the point $(x,b)$ has distance 1 from the points $(a,b+1)$, $(a+1,b+1)$, and $(x-1,b)$ and so must have the colour 3.

Similarly, for $a-1\le x<a-d$ the point $(x,b)$ has colour 1.

Any point $(x,b+1)$ with $a-d<x<a+1-d$ must then have the colour 4.

But now we can consider also points $(x,b)$ with $a-1-d<x<a-1$, which must then have the colour 1. Similarly points $(x,b)$ with $a+1<x<a+2-d$ must have the colour 3.

Obviously, this type of argument shows, by induction on $|m|$, that for all integers $m$ and for $n\in\{-1,0,1\}$ the colour of a point $(a+n+x,b+m)$, where $-d<x<1-d$, is the same as that of the point $(a+n,b+m)$.

Actually, this last statement is valid for all integers $n$.

To see this it suffices to consider the case $n=-2$, $m=0$. The hitch is that the colour of $(a-2,b)$ can be chosen freely from the set $\{2,4\}$. But in any case, the colour of $(a-2+x,b)$, where $-d<x<1-d$, must be different from those of the points $(a-1+x,b)$, $(a-1+x,b+1)$, and $(a-2,b+1)$, which leaves the colour of $(a-2,b)$ as the only possibility, q.e.d.

It also follows that after we for each point $(a+n,b)$, where the integer $n$ satisfies $|n|>1$, have chosen an admissible colour (i.e. of the same parity as $n$), the colouration of all lines $y = b + m$ ($m$ integral) is determined.

The next step is to consider the colouration of the lines $y = b + u$ where $0 < u < 1$. We can see that the colour of a point $(a + n + x, b + u)$, where $-d < x < 1 - d$, must have the parity of $n$. This implies that the colour for fixed values of $n$ and $u$ does not depend on $x$ for $-d < x < 1 - d$, since otherwise we have a contradiction when trying to colour points on the line $y = b + u + 1$.

Now is the time to introduce some simplification.

We have already used an isometry to arrive at the supremum norm case. Let us further use translation, possibly combined with a change of direction of the $x$-axis, to get to the situation, where $a = b = d = 0$, and the points $(x, 0)$ with $0 \le x < 1$ have the colour 2.

For each $u \in [0, 1)$ we choose a sequence $(C_n(u) \mid -\infty < n < \infty)$ of colours satisfying the condition that for all $n$ the colour $C_n(u)$ has the parity of $n$ (we have already chosen $C_0(0) = 2$).

Then the colouration is completely described by the following:

For each point $(x, y)$ we define $C(x, y) = C_{[x]}(y - [y])$ if $[y]$ is even, otherwise $C(x, y) = C_{[x]}(y - [y]) \pm 2$ (exactly one of these two numbers belongs to $M$).

Finally we must check that the colouration found is also admissible:

Let the two points $(x_1, y_1)$ and $(x_2, y_2)$ have distance 1. Then, if $[x_1] = [x_2]$, we must have $|y_1 - y_2| = 1$, and $|C(x_1, y_1) - C(x_2, y_2)| = 2$. Otherwise $|[x_1] - [x_2]| = 1$, and $C(x_1, y_1)$ and $C(x_2, y_2)$ have different parity.

We have shown

**Theorem 2.** *Each permissible four-colouring of the plane with the supremum norm and with a horizontal or vertical line which is not two-coloured, is equivalent to a four-colouring constructed as follows:*

*For each $u \in [0, 1)$ we choose a sequence $(C_n(u) \mid -\infty < n < \infty)$ of colours satisfying the condition that for all $n$ the colour $C_n(u)$ has the parity of $n$.*

*For each point $(x, y)$ we define the colour as*

$$C(x, y) = \begin{cases} C_{[x]}(y - [y]) & \textit{if } [y] \textit{ is even} \\ C_{[x]}(y - [y]) \pm 2 & \textit{otherwise.} \end{cases}$$

*This paper was slightly modified on 5 December 2014.*

## References

1. N. G. de Bruijn and P Erdős, *A colour problem for infinite graphs and a problem in the theory of relations*, Nederl. Akad. Wetensch. Indag. Math. **13** (1951), 371–373.
2. K. B. Chilakamarri, *The unit-distance graph problem: a brief survey and some new results*, Bulletin of the ICA **8** (1993), 39–60.

Fjordparken 23A, DK–4000 Roskilde, Danmark
*E-mail address*: gkkristi@post6.tele.dk