ON THE DISTRIBUTION MODULO ONE OF GEOMETRIC SEQUENCES

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ABSTRACT. An important property of a geometric sequence is its asymptotic distance from the integers. The infimum of these distances for all geometric sequences with the same ratio $\alpha > 1$ is a property $C(\alpha)$ of the ratio. Estimates of $C(\alpha)$ are given, depending on the algebraic nature of the number α.

INTRODUCTION

For all real numbers x we define $\|x\|$ as the distance from x to the integers. Next, the function $C : (1, \infty) \to [0, \frac{1}{2}]$ is defined as

$$C(\alpha) = \inf_{\lambda>0} \limsup_{n\to\infty} \|\lambda\alpha^n\|. \tag{1}$$

Obviously, C takes the value 0 on the integers.

Here, and in the following example, the infimum in (1) is obtained with $\lambda = 1$.

If $\alpha > 1$ is a zero of a monic irreducible polynomial P with integral coefficients, and its conjugates (the remaining zeros of P) have absolute value less than one, α is called a PV-number, or a Pisot-number, or is said to belong to the set S. Then $C(\alpha) = 0$, simply because the sum of the nth powers of the zeros of P is an integer, and the contribution from the zeros different from α tends towards zero when $n \to \infty$.

If $\alpha > 1$ is a zero of a monic irreducible polynomial P with integral coefficients, and its conjugates have absolute value less than or equal to one (at least one, and then, in fact, all but one of them, having absolute value equal to one), α is called a Salem-number, or is said to belong to T. What can be said about $C(\alpha)$ in this case?

We first note that the set of zeros of P is the union of 2-sets $\{w_1, w_2\}$ with $w_1w_2 = 1$. One such set is $\{\alpha, 1/\alpha\}$, while the other 2-sets have the form $\{z_j, \tilde{z}_j\}$, where $j = 1, \ldots, k$ and, for definiteness, $0 < \theta_j = \arg z_j < \pi$ for all j. The degree of P is then $2k + 2$.

We shall show that the numbers 1 and $\theta_j/(2\pi)$ $\quad (j = 1, \ldots, k)$ are linearly independent over the integers. Otherwise we could find integers t_j $\quad (j = 0, \ldots, k)$, not all zero, such that

$$\sum_{j=1}^{k} t_j\theta_j = 2t_0\pi, \tag{2}$$

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i.e.

$$\prod_{j=1}^{k} z_j^{t_j} = 1. \tag{3}$$

For definiteness, we assume $t_1 \neq 0$.

Let K be a splitting field of P. Let σ be an automorphism of K satisfying $\sigma(z_1) = \alpha$. Since $z_j \neq \tilde{z_1}$ for $j \neq 1$ we get the contradictory $\alpha^{t_1} = 1$ when taking the absolute value of the transformation of (3) by σ.

An application of Kronecker's Theorem now shows that for arbitrary real numbers $r_1, \ldots, r_k$ and arbitrary real $\epsilon > 0$ there are an infinite number of integers n such that integers $p_1, \ldots, p_k$ exist, for which

$$|n\theta_j - 2p_j\pi - r_j| < \epsilon \tag{4}$$

for $j = 1, \ldots, k$.

From (4) follows, in particular, that the numbers $\| \sum_{j=1}^{k} 2\cos n\theta_j \|$ are dense in $[0, \frac{1}{2}]$. The same must then be true for $\|\alpha^n\|$. Clearly, $\lambda = 1$ is not an interesting choice. But I cannot exclude the possibility that it might still be possible to obtain a general estimate of $C(\alpha) < \frac{1}{2}$ for these numbers.

To my knowledge, for no other number α than those already mentioned the exact value of $C(\alpha)$ is known. I shall give upper and lower bounds for $C(\alpha)$ obtainable by simple methods.

General bounds

If nothing is known about the algebraic nature of α, we can only give rather coarse bounds for $C(\alpha)$.

The following results are implicit in [3, Chapter II]:

For all $\alpha > 2$ we have $C(\alpha) \leq \frac{1}{2(\alpha-1)}$.

For all $\alpha > 1$, except for a countable set, we have $C(\alpha) \geq \frac{1}{2(\alpha+1)^2}$. Thus, at a point where $C(\alpha) = 0$, C cannot be upper semicontinuous.

If α belongs to a certain countable dense subset of $(1, \infty)$ (the so-called E-numbers), we have $C(\alpha) \leq \frac{1}{2(\alpha-1)^2}$. Those E-numbers which do not belong to $S \cup T$ are generally believed to be transcendental numbers (see [1]).

The ratio is an algebraic integer

Assume that $\alpha > 1$ is a zero of a monic polynomial P with integral coefficients, and that all the other zeros of P also have absolute value greater than one. We also assume that α is not a natural number. Let $\lambda > 0$ and put, for $n \in \mathbb{N}_0$,

$$\lambda\alpha^n = p_n + r_n, \tag{5}$$

where $p_n \in \mathbb{Z}$, and $|r_n| = \|\lambda\alpha^n\|$.

Let $P(z) = \sum_{j=0}^{m} a_j z^j$, where $a_m = 1$. Then, using (5), for $n \in \mathbb{N}_0$,

$$0 = \sum_{j=0}^{m} \lambda a_j \alpha^{n+j} = \sum_{j=0}^{m} a_j p_{n+j} + \sum_{j=0}^{m} a_j r_{n+j}. \tag{6}$$

If now $|r_n| < \frac{1}{\sum_{j=0}^{m} |a_j|}$ for $n \geq n_0$, we must have

$$\sum_{j=0}^{m} a_j r_{n+j} = 0 \quad \text{for } n \geq n_0. \tag{7}$$

But then, because of the assumption concerning the zeros of P, we shall have $|r_n| \to \infty$ for $n \to \infty$, unless $r_n = 0$ for all but a finite number of n-values. The first case cannot occur, and the second is also excluded, since we have assumed that α is not a rational integer.

We conclude that in this case $C(\alpha) \geq \frac{1}{\sum_{j=0}^{m} |a_j|}$.

Example: $\alpha = \sqrt{n}$, where n is a natural number and not a square. Then we see that $C(\sqrt{n}) \geq \frac{1}{n+1}$. In particular, the function C cannot be lower semicontinuous at such a point $\alpha = \sqrt{n}$ when $n \geq 14$. In fact, in any neighbourhood of such a point there are E-numbers, and so there is a sequence (α_n) of numbers approaching $\sqrt{n}$, so that $\limsup_{n\to\infty} C(\alpha_n) \leq \frac{1}{2(\sqrt{n}-1)^2}$. But $2(\sqrt{n}-1)^2 > n+1$ for $n \geq 14$.

The ratio is a non-integral algebraic number

Assume that $\alpha > 1$ is a zero of a primitive irreducible, but not monic, polynomial P with integral coefficients. Let $P(z) = \sum_{j=0}^{m} a_j z^j$, where $a_m > 1$. Eqs.(5) and (6) are again valid. But then, if $|r_n| < \frac{1}{\sum_{j=0}^{m} |a_j|}$ for all $n \geq n_0$, Eq.(7) is also valid for these n-values. We also have $\sum_{j=0}^{m} a_j p_{n+j} = 0$ for $n \geq n_0$. The generating function

$$f(z) = \sum_{n=0}^{\infty} p_n z^n \tag{8}$$

is then a rational function

$$f(z) = \frac{Q(z)}{P_1(z)}, \tag{9}$$

where P_1 is the polynomial reciprocal to the polynomial P defined above: $P_1(z) = z^m P(\frac{1}{z}) = \sum_{j=0}^{m} a_{m-j} z^j$. Q has integral coefficients, and Q and P_1 have no common factors. But then, according to a theorem of Fatou (see [4, page 4]), $P_1(0) = \pm 1$, contradicting $a_m > 1$.

We conclude that also in this case $C(\alpha) \geq \frac{1}{\sum_{j=0}^{m} |a_j|}$.

The ratio is half an odd integer

Let $\alpha = \frac{p}{2}$, where p is an odd integer at least equal to 3. From the above we have

$$\frac{1}{p+2} \leq C(\alpha) \leq \frac{1}{p-2}.$$

We can improve both inequalities.

We again use the notation of Eq.(5), which gives us,

$$p_{n+1} - \alpha p_n = \alpha r_n - r_{n+1}. \tag{10}$$

We first show that $C(\frac{p}{2}) \leq \frac{1}{p}$.

Let $p_0 \in \mathbb{N}$ be given. We shall show that there is a $\lambda \in (p_0, p_0+1)$ so that $\|\lambda(\frac{p}{2})^n\| \leq \frac{1}{p}$ for all $n \in \mathbb{N}_0$.

We first generate the sequence (p_n) of integers and an auxiliary sequence (ϵ_n), where for all $n \in \mathbb{N}_0$ we have $\epsilon_n \in \{-1, 0, 1\}$, by a method similar to the one used in [2].

In step n the numbers p_m and ϵ_m for $m < n$ are given.

We put $\epsilon_n = 0$ if p_{n-1} is even. If $n-1$ is the lowest suffix for which p_{n-1} is odd, we put $\epsilon_n = 1$. In the other cases where p_{n-1} is odd we put $\epsilon_n = -\epsilon_m$, where m is the maximal suffix satisfying both $m < n$ and $\epsilon_m \neq 0$.

When ϵ_n is determined, we put

$$p_n = \frac{pp_{n-1} + \epsilon_n}{2}. \tag{11}$$

When the sequence (ϵ_n) is generated, we define, for all $n \in \mathbb{N}_0$,

$$r_n = \sum_{k=1}^{\infty} \left(\frac{2}{p}\right)^{k-1} \frac{\epsilon_{n+k}}{p}. \tag{12}$$

Note that the signs of the non-zero terms on the rhs of (12) alternate and that these terms decrease in absolute value when k increases. Thus r_n has the sign of ϵ_m, where m is the minimal suffix satisfying both $m > n$ and $\epsilon_m \neq 0$. In particular, $r_0 > 0$. Moreover, for all $n \in \mathbb{N}_0$, $|r_n| < \frac{1}{p}$.

Putting $\lambda = p_0 + r_0$, we have $p_0 < \lambda < p_0 + \frac{1}{p} < p_0 + 1$. Eq.(12) implies that $r_n = \frac{\epsilon_{n+1}}{p} + \frac{2}{p} r_{n+1}$ for all $n \in \mathbb{N}_0$. But then Eq.(10) is satisfied for all $n \in \mathbb{N}_0$, and so is Eq.(5). Thus we have indeed shown that $C(\frac{p}{2}) \leq \frac{1}{p}$.

Next we show that $C(\frac{p}{2}) \geq \frac{p}{p^2+4}$.

Let $\frac{1}{p+2} < \delta < \frac{p}{p^2+4}$. We shall derive a contradiction from the assumption that for some $\lambda > 0$ we should have $\|\lambda(p/2)^n\| \leq \delta$ for all $n \in \mathbb{N}_0$.

With the notation of (10) there are only two possibilities for the relation between $|r_n|$ and $|r_{n+1}|$:

Either

p_n is even, r_n and r_{n+1} have the same sign, and $|r_{n+1}| = (p/2)|r_n|$ (note that $|r_n| < 1/p$),

or

p_n is odd, and (10) gives $|(p/2)r_n - r_{n+1}| = \frac{1}{2}$. Thus r_n and r_{n+1} have opposite signs, and

$$|r_{n+1}| = (1 - p|r_n|)/2. \tag{13}$$

Clearly, p_n must be odd for some n, say for $n = n_0$. Then (13) should be used to determine $|r_{n+1}|$. Now $|r_n| \leq \delta$ implies $|r_{n+1}| \geq (1-p\delta)/2$. But if p_{n+1} is even, we must have $|r_{n+1}| \leq 2\delta/p$, since, otherwise, we would have $|r_{n+2}| > \delta$. And so we should have $(1-p\delta)/2 \leq 2\delta/p$, i.e. $\delta \geq \frac{p}{p^2+4}$, contrary to assumption.

Thus p_n is odd, and (13) is fulfilled, for $n \geq n_0$.

We rewrite (13) as

$$|r_{n+1}| - \frac{1}{p+2} = -\frac{p}{2}\left(|r_n| - \frac{1}{p+2}\right),$$

and so

$$|r_n| - \frac{1}{p+2} = \left(-\frac{p}{2}\right)^{n-n_0} \left(|r_{n_0}| - \frac{1}{p+2}\right).$$

But $|r_n|$ is bounded, and so we must have $|r_n| = 1/(p+2)$ for $n \geq n_0$.

Since r_n and r_{n+1} have opposite signs, (10) gives

$$p_{n+1} = \frac{pp_n + \epsilon_n}{2} \quad [n \geq n_0), \tag{14}$$

where $\epsilon_n = \text{sign}(r_n)$.

Equation (14) can be written as

$$(p+2)p_{n+1} + \epsilon_{n+1} = \frac{p}{2}\left((p+2)p_n + \epsilon_n\right),$$

or

$$(p+2)p_n + \epsilon_n = \left(\frac{p}{2}\right)^{n-n_0} \left((p+2)p_{n_0} + \epsilon_{n_0}\right)$$

for all $n \geq n_0$, which is impossible, since the integer $(p+2)p_{n_0} + \epsilon_{n_0}$ cannot be divisible by arbitrarily high powers of 2.

Thus, we must have $C(\frac{p}{2}) \geq \frac{p}{p^2+4}$, q.e.d.

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