ENTIRE FUNCTIONS

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Introduction.

I published two types of papers where results for polynomials were generalized to entire functions.

First, based on the earlier mentioned observation that with the employed variational method the extremizing function was characterized by means of its stationary values, it was natural to find out to which degree a function could, in fact, be determined in this way.

Next, in a number of papers I had studied a function, which I called Kuhn's ratio, and which was associated with an (algebraic or trigonometrical) polynomial. Such a function could also be associated with an entire function provided the latter satisfied a few conditions, and some of the properties of Kuhn's ratio could also be extended.

Stationary values.

The most general theorem I proved (see [1]) associated with an (almost) arbitrary ordered sequence of real numbers (each associated with a natural number) a real function whose stationary values were the members of the sequence taken on in the given order, and for each member with the associated natural number as multiplicity. The "almost" means that the parity of the multiplicity of the stationary value must be compatible with the signs of the change of the function to the left and right of the point we consider. The derivative of the function is a Pólya-Laguerre function with all zeros real.

The function itself is uniquely determined apart from an arbitrary real affine transformation of the independent variable.

After rereading my proof I found it natural to add explanatory notes concerning two places in the paper:

(1) p. 278 (first half of the page): The function $\phi = g^{-1} \circ f$ is defined, first in a vicinity of the real axis, next along the later cuts, and finally as $(\log g)^{-1} \circ \log f$ in the cut plane.

(2) p. 293 (concerning the sequence $(x_j^{(p)})$ for $j < 0$). We have $x_j^{(p)} \geq x_j^{(p+1)}$. To show that the sequence does not diverge towards $-\infty$, we first transform f_p to $P_{p,q}$ by increasing $|y_q|$ to infinity (which will decrease x_j even more), next transform f_{-q} to $P_{p,q}$ by increasing $|y_{-p-1}|$ to infinity, which will increase the (already converged) x_j. Thus we have a lower bound for the sequence, which must then converge.

I was later informed by Professor Sodin that this theorem was "well known" (unfortunately neither by me nor by the referee) and had been proved 47 years earlier by Gerald MacLane (I knew of MacLane's more recent work but had not been able to extrapolate back to the paper in question). His method was considerably simpler than mine, and also other (Russian) authors had published simple proofs of the theorem.

Anyway, this information dispelled all doubts I might have had with respect to the validity of the theorem, which was a relief.

Incidentally, Professor Bo Kjellberg helped me a lot with these papers. He also commented on those already written. For instance, my one-page proof on pages 279 and 280 of [3] could be reduced to the following few lines:

Starting from equation (2) of [3] and noting that $\log(1+t) < t$ for $t > 0$, we consider the two cases:

(1) There is an N so that

$$\sum_{n=1}^{N} \frac{1}{a_n} > b.$$

Then

$$\log f(x) < d + x(b - \sum_{n=1}^{N} \frac{1}{a_n}) + \sum_{n=1}^{N} \log(1 + \frac{x}{a_n}) \xrightarrow[x \longrightarrow \infty]{} -\infty.$$

(2) We have

$$\sum_{n=1}^{\infty} \frac{1}{a_n} \leq b.$$

Then

$$\log f(x) = \lim_{N \longrightarrow \infty} \left(d + x(b - \sum_{n=1}^{N} \frac{1}{a_n}) + \sum_{n=1}^{N} \log(1 + \frac{x}{a_n}) \right) \geq d + \log(1 + \frac{x}{a_1}) \xrightarrow[x \longrightarrow \infty]{} \infty.$$

Kuhn's ratio.

For real entire functions f mapping $\mathbb{R}$ into $[-1,1]$ I define Kuhn's ratio as

$$R_f(x) = (1 - f^2(x))/(f'(x))^2$$

(the substitution $t = \cos x$ gives a definition of Kuhn's ratio also for algebraic polynomials mapping $[-1,1]$ into itself).

I proved that R_f is logarithmically convex at its real points of regularity if only f belongs to the Pólya-Laguerre class. See [4].

References

1. G. K. Kristiansen, *On the existence of real entire functions with a prescribed ordered set of stationary values*, Journal of Approximation Theory **75** (1993), 266–294.
2. ———, *Erratum*, Journal of Approximation Theory **80** (1995), 150.
3. ———, *Characterization of real entire functions by means of their stationary values*, Arch. Math. **56** (1991), 278–280.
4. ———, *A property of functions of the Pólya-Laguerre class.*, www.kristiansensmathematics.dk/unpublished.