CONNES'S WORK ON ZEROS OF POLYNOMIALS

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Abstract. This is a personal version of Connes's results. I have tried to complete the proofs.

Introduction

The work referred to is contained in the three papers ([1], [2], and [3]), of which [3] is the only one containing something resembling proofs.

Polynomials with all zeros outside the unit circle

This section is based on [3, page 18-01].

Let $P(x) \in \mathbb{C}[x]$ have all zeros outside the unit circle.

We consider the set of submultiplicative seminorms $z \mapsto \|z\|$ on the algebra $A = \mathbb{C}[x]/P(x)$.

Notationally we shall not distinguish between a polynomial in $\mathbb{C}[x]$ and its canonical image in A. However, it is convenient to distinguish between the scalar (i.e. the complex number) 1 and the monomial e, whose set of values equals $\{1\}$.

An obvious consequence of the submultiplicity is the fact that if there is an element $z \in A$ with $\|z| \neq 0$, i.e. if the seminorm is non-trivial, then $\|e\| \geq 1$.

Moreover, if the seminorm is non-trivial, its zero-set is an ideal generated by a polynomial Q, which must divide P.

Let Q have a zero α. Put $Q(x) = (x-\alpha)Q_1(x)$. Then

$$0 = \|Q(x)\| = \|(x-\alpha)Q_1(x)\|,$$

so that

$$|\alpha|\|Q_1(x)\| = \|xQ_1(x)\| \leq \|x\|\|Q_1(x)\|.$$

Since $\|Q_1(x)\| > 0$, we obtain $\|x\| \geq |\alpha| > 1$.

A submultiplicative seminorm is defined by

$$\|z\| = \inf\{\sum_{j=0}^{n} |b_j| \mid z = \sum_{j=0}^{n} b_j x^j\}. \tag{1}$$

For the monomial x we find $\|x\| \leq 1$, which in combination with the remarks above shows that (1) is trivial.

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In particular, $\|e\| = 0$, and so there exists a polynomial $\sum_{j=0}^{n} a_j x^j$ which is a multiple of $P(x)$ so that $|1 - a_0| + \sum_{j=1}^{n} |a_j| < 1$. Thus,

$$|a_0| \geq 1 - |1 - a_0| > \sum_{j=1}^{n} |a_j|. \tag{2}$$

Conversely, if $P(x)$ has a zero θ with $|\theta| \leq 1$, for every multiple $\sum_{j=0}^{n} a_j x^j$ of $P(x)$ we have

$$-a_0 = \sum_{j=1}^{n} a_j \theta^j,$$

implying

$$|a_0| \leq \sum_{j=1}^{n} |a_j|, \tag{3}$$

making the condition that some multiple $\sum_{j=0}^{n} a_j x^j$ of $P(x)$ satisfies (2) a necessary and sufficient for $P(x)$ to have all zeros outside the unit circle.

Weak orderings of fields finite over $\mathbb{Q}$

Let K be a field finite over $\mathbb{Q}$. Its elements are algebraic numbers.

A weak partial ordering ω is defined by the corresponding set of positive numbers (also denoted by ω). This set must satisfy the following requirements (and only these):

(1) Its intersection with $\mathbb{Q}$ does not contain the number 0.
(2) If x and y are members of ω, so are all numbers $\lambda x + \mu y$, where λ and μ are positive rational numbers.
(3) If x and y are members of ω, so is the number $1/(1/x + 1/y)$.

For simplicity, the statement $x > 0$ will be considered synonymous to $x \in \omega$.

Assume that we are given n numbers ω_j ($j = 0, 1, \ldots, n-1$) in ω with the property that their rational linear combinations span the module $L = \omega - \omega$. We shall assume that these n numbers form a subset of ω which is maximal with respect to the property of being linearly independent over $\mathbb{Q}$.

Any number $x \in L$ can be written $x = \sum_{j=0}^{n-1} \xi_j \omega_j$, where $\forall j : \xi_j \in \mathbb{Q}$.

Put $z = \sum_{j=0}^{n-1} \omega_j$. Then $z > 0$, and for any $x \in L$ there exists a $\lambda \in \mathbb{Q}_+$, such that $\lambda z > x$.

We shall show that $L = zK'$, where K' is a subfield of K.

We consider $\omega' = \omega/z$, which is a weak partial ordering iff ω is.

And so

$$x \in \omega' \implies \frac{1}{\frac{1}{x} + \frac{1}{\lambda - x}} = x(1 - x/\lambda) \in \omega',$$

i.e. $x^2 \in L' = \omega' - \omega'$.

Let $x \in \omega'$, $y \in \omega'$. Then

$$xy = \frac{1}{2}((x+y)^2 - x^2 - y^2) \in L'. \tag{4}$$

Hence, $x \in L', y \in L' \implies xy \in L'$. This means that the $\mathbb{Q}$-module L' is in fact a ring. Since L' is a subring of the field K and of finite dimension over the field $\mathbb{Q}$, it is easy to show that L' is itself a field (in fact, the inverse of an element $x \in L'$ belongs to $\mathbb{Q}[x]$ and so to L').

We shall show that a given partial weak ordering ω of K can always be completed to a total weak ordering of K.

However, it follows from the above that the real problem is to complete the partial weak ordering ω' of $L' = K'$ to a total weak ordering of K'. So, in the following we shall simply remove the apostrophes and directly tackle the problem of extending the given partial ordering ω of $L = K = \mathbb{Q}[x]/p$, where $p \in \mathbb{Q}[x]$ is irreducible.

Let the zeros of p be θ_j $(j = 1, \ldots, r + 2c)$, where θ_j is real for $j = 1, \ldots, r$ and has positive imaginary part for $j = r + 1, \ldots, r + c$, while $\theta_{r+c+k} = \widetilde{\theta}_{r+k}$ for $k = 1, \ldots, c$.

Each element q of K can be written as a polynomial of degree at most $n - 1$, where $n = r + 2c$ is the degree of p. Each isomorphism ϕ of K into $\mathbb{C}$ has the form $\phi = \phi_j$, where $\phi_j(q) = q(\theta_j)$ for some $j \in \{1, \ldots, n\}$.

It is practical to collect these isomorphisms into a single isomorphism ϕ of K into the ring $A = \mathbb{R}^r \times \mathbb{C}^c$. Here we have used the fact that q is a real polynomial, so that only the $r + c$ first isomorphisms are needed to characterize q.

Then q can be found by means of the Lagrange interpolation formula, for instance, calculating $q(\theta_{r+c+k})$ as the complex conjugate of $q(\theta_{r+k})$ for $k = 1, \ldots, c$.

This also shows that $\phi(K)$ is dense in A. In fact, to each point in A we can find a polynomial $q_0 \in \mathbb{R}[x]$ of degree at most $n - 1$ corresponding to it. But q_0 can be arbitrarily well approximated by polynomials in $\mathbb{Q}[x]$.

The set $\phi(\omega)$ gives a partial ordering of A.

Consider the images $\omega_j = \phi_j(\omega)$ for $j = 1, \ldots, r + c$.

In the real case $(j \leq r)$ the closure of ω_j must be either one of the semiaxes $[0, \pm\infty)$ or the whole of $\mathbb{R}$; this follows directly from condition (2) for ω.

In the non-real case $(r + 1 \leq j \leq r + c)$ the angle of the smallest closed cone containing ω_j either is at most π, or is equal to 2π, because of condition (2).

So a natural first step towards a total ordering of K is to replace $\phi(\omega)$ by the interior ω_A of its completion in A. This is again a partial weak ordering of A. Note, in particular, that although 0 belongs to the completion of $\phi(\omega)$ in A, it is a boundary point and so does not belong to ω_A.

More information about ω_A gives Connes's

Lemma ([3, p. 18-08])**.** *The partial ordering ω_A does not depend on those coordinates for which the projection of ω has as closure the whole field $\mathbb{C}$ or $\mathbb{R}$.*

Proof. Assume, for definiteness, that ω_A contains the points x_1 with first coordinate ξ and x_2 with first coordinate $-\xi$, chosen such that the other coordinates of both x_1, x_2 and $x_1 + x_2$ are non-zero. Then ω_A contains also the limit for $\epsilon \to 0$ of the expression

$$(2\epsilon)/((1 + \epsilon)/x_1 + (1 - \epsilon)/x_2),$$

which is the point with first coordinate ξ and the other coordinates equal to 0. In fact, all points with the first coordinate a real multiple of ξ and the $(n - 1)$-tuple of the remaining coordinates a non-negative linear combination of those of x_1 and x_2 are contained in ω_A.

Consider the case where there is no value of j, for which ω_j is restricted to a semiaxis or a halfplane. Then the closure of $\phi(\omega)$ is the whole of A, in particular, 0 should be an interior point in this closure and so belong to ω_A, which is impossible.

Thus there is at least one number j, for which $\overline{\omega_j}$ is not the whole field. Let $j = j_0$ be such a number.

There are three cases:

1) If $j_0 \leq r$, we have $\overline{\omega_{j_0}}$ equal to either $[0, +\infty)$ or $[0, -\infty)$, and we can simply extend ω_A by replacing it with the product $\mathbb{R}^{j_0-1}\mathbb{R}_+\mathbb{R}^{r-j_0}\mathbb{C}^c$ or the one obtained by replacing $\mathbb{R}_+$ with $\mathbb{R}_-$ in this expression.

If $j_0 > r$, we first determine an open halfplane Π containing the interior of $\overline{\omega_{j_0}}$. Then we introduce two real rectangular coordinates x_{j_0} and y_{j_0}, such that Π equals $x_{j_0} > 0$. If now

2) ω_{j_0} has points on both of the y_{j_0}-semiaxes, the lemma shows that ω_A does not depend on y_{j_0}, and we just extend ω_A by replacing it with the product

$$\mathbb{R}^r\mathbb{C}^{j_0-r-1}\Pi\mathbb{C}^{r+c-j_0}.$$

The same applies if ω_{j_0} has points on none of the y_{j_0}-semiaxes.

If, however,

3) ω_{j_0} has points on at most one of the semiaxes from 0 to ∞ bounding Π, the product $\mathbb{R}^r\mathbb{C}^{j_0-r-1}\Pi\mathbb{C}^{r+c-j_0}$ does not suffice as extension of ω_A. We will have to add a set containing the remaining points of ω_A. To do this we consider the other coordinates of those points of ω_A, for which coordinate number j_0 belongs to the relevant semiaxis $\mathbb{R}_\pm$. A new coordinate j_1 must receive the same treatment as j_0 above.

We find that the total weak orderings ω continuing the given one are lexographical corresponding to some ordering of k (say) of the non-real coordinates, according to the following scheme:

An element $x \in K$ is positive iff one of the following possibilities occur,

1) for some $m < k$: $x_{j_0}(x) = \cdots = x_{j_{m-1}}(x) = 0, x_{j_m}(x) > 0$,

2) $x_{j_0}(x) = \cdots = x_{j_k}(x) = 0, \pm x_{j_{k+1}}(x) > 0$,

where the coordinate with number j_{k+1} is real, and the appropriate sign is taken.

Applications of weak ordering

See [3, pages 18-07 and 18-08].

Let $x_1, \ldots, x_n$ be variables. Define $\Omega(x_1, \ldots, x_n)$ as the set of rational functions obtained from $x_1, \ldots, x_n$ by applying the procedures (2) and (3) for positive sets a finite number of times.

Let $u_1, \ldots, u_n$ be algebraic numbers. They can be represented by polynomials in some field $K = \mathbb{Q}[x]/q$, where $q \in \mathbb{Q}[x]$ is irreducible.

The conjugate systems to $(u_1, \ldots, u_n)$ will be denoted by $(u_1^{(p)}, \ldots, u_n^{(p)})$.

Theorem 1. *A necessary and sufficient condition for the existence of a function $F \in \Omega(x_1, \ldots, x_n)$ such that $F(u_1, \ldots, u_n) = 0$ is that, for every p, 0 belongs to the (real) convex hull of $u_1^{(p)}, \ldots, u_n^{(p)}$.*

Proof. If $F(u_1, \ldots, u_n) = 0$, then also, for all p, $F(u_1^{(p)}, \ldots, u_n^{(p)}) = 0$, and so 0 belongs to the convex hull of $u_1^{(p)}, \ldots, u_n^{(p)}$.

Conversely, assume that for all $F \in \Omega(x_1, \dots, x_n)$ we have $F(u_1, \dots, u_n) \neq 0$, and let ω be a set containing $u_1, \dots, u_n$. Then $F(u_1, \dots, u_n) \in \omega$ for all $F \in \Omega(x_1, \dots, x_n)$. But then, according to our assumption, ω satisfies all conditions necessary to qualify it as a weak partial ordering. And so, as we have seen, there is a p for which ω_p is contained in an open halfplane. And then 0 does not belong to the convex hull of the $u_1^{(p)}, \dots, u_n^{(p)}$, q.e.d.

Examples:

1) A polynomial in $\mathbb{Q}[x]$ whose zeros are all real and negative divides a rational function $F(x, 1)$, where $F \in \Omega(x_1, x_2)$.

2) A polynomial in $\mathbb{Q}[x]$ whose zeros all have negative real part divides a rational function $F(x, 1/x, 1)$, where $F \in \Omega(x_1, x_2, x_3)$.

3) Let $P_1, \dots, P_k$ be points in $\mathbb{C}$, algebraically members of the field $\kappa = \mathbb{Q}[i]$ and geometrically corners in a convex polygon $P_1 \dots P_k$.

Then a complex number z is a zero of a polynomial in $\kappa[x]$ with all zeros in the given polygon iff there is a function $F \in \Omega(x_1, \dots, x_k)$ such that $F(z - P_1, \dots, z - P_k) = 0$.

In fact, a conjugate of the k-tuple $(z - P_1, \dots, z - P_k)$ either has the form $(z_j - P_1, \dots, z_j - P_k)$ (where z_j is another zero of the equation satisfied by z) or the form $(\tilde{z}_j - \tilde{P}_1, \dots, \tilde{z}_j - \tilde{P}_k)$.

4) Assume now that $u_1, \dots, u_n$ all belong to the same open halfplane, but that this is not true for any other $u_1^{(p)}, \dots, u_n^{(p)}$ except $\tilde{u}_1, \dots, \tilde{u}_n$.

Let γ denote the smallest closed cone containing the origin and $u_1, \dots, u_n$.

Let $G(u_1, \dots, u_n)$ be a rational homogeneous function of degree 1 with rational coefficients and with its value contained in γ. Then $G(u_1, \dots, u_n)/u_1$ is a member of the field $\omega - \omega$, where ω is an arbitrary weak partial ordering containing $1, u_2/u_1, \dots, u_n/u_1$, in fact even a member of ω (note that because of our assumptions about $u_1^{(p)}, \dots, u_n^{(p)}$ it is not necessary to consider the other projections). Defining ω as the set of values $F(1, u_2/u_1, \dots, u_n/u_1)$ obtained by letting $F(x_1, \dots, x_n)$ run through $\Omega(x_1, \dots, x_n)$, we find that there is a function $F(x_1, \dots, x_n)$ in $\Omega(x_1, \dots, x_n)$, such that $G(u_1, \dots, u_n) = F(u_1, \dots, u_n)$.

Example:

If θ is a positive algebraic number whose conjugates all have negative real part, we can put $(u_1, u_2, u_3) = (\theta, 1, 1/\theta)$ and use the result just proved: We can choose positive rational numbers λ_1 and λ_2, such that the algebraic number u, defined by

$$\theta = \lambda_1 \times 1 + \frac{1}{\lambda_2 \times 1 + 1/u}$$

is positive. Then u is a function of the type G above, and as u is positive, we conclude that there is a function $F \in \Omega(x_1, x_2, x_3)$ such that $u = F(\theta, 1, 1/\theta)$. Thus,

$$\theta = \lambda_1 + \frac{1}{\lambda_2 + 1/F(\theta, 1, 1/\theta)}. \tag{12}$$

The case where θ is a positive algebraic number whose conjugates all are negative real can be treated similarly: We have $\theta = F(\theta, 1)$, where $F \in \Omega(x_1, x_2)$.

Rational polynomials with exactly one zero inside the unit circle

Theorem 2. *A necessary and sufficient condition for θ to be an algebraic number with $|\theta| < 1$ and with all conjugates outside the unit circle is that there are rational numbers $a_0, \dots, a_n$ such that $\sum_{j=0}^{n} |a_j| < 1$ and*

$$\theta = \sum_{j=0}^{n} a_j \theta^j .$$

One may choose the a_j such that $a_1 = 0$ and only p of them are different from 0. Here p is the number of conjugates of θ.

Proof. The sufficiency is obvious (Rouché).

To prove the necessity, let $q \in \mathbb{Q}[x]$ be the irreducible polynomial of which θ is a zero.

As above we consider the field $K = \mathbb{Q}[x]/q$ and its isomorphisms into $\mathbb{C}$.

As in the first section (but with $\mathbb{C}[x]$ replaced with $\mathbb{Q}[x]$) we can define the seminorm (1).

As in the second section we consider the isomorphism ϕ of K into the ring $A = \mathbb{R}^r \times \mathbb{C}^c$, where r is the number of real zeros and c the number of zeros with positive imaginary part. We have $r + 2c = p + 1$.

But clearly the isomorphism ϕ is the restriction of a homomorphism of $\mathbb{C}[x]/q$ onto A, defined nominally in the same way. Similarly for the seminorm (1) (where now n is an unlimited positive integral parameter), since $\mathbb{Q}$ is dense in $\mathbb{C}$.

We can consider the extended seminorm as a seminorm on A. The set of elements of A on which the seminorm vanishes is an ideal and so is characterized by a set of coordinates vanishing. To find out what this set is, we consider the unit ball $B \subset K$. A linear form vanishes on all elements with seminorm zero iff the image of B is bounded. Thus the seminorm vanishes on the coordinate corresponding to a particular zero of q iff this zero has absolute value less than or equal to 1. But this means that the seminorm does not depend on the other coordinates. In the case we consider, we must have $\|p\| = k|p(\theta)|$. Since $\|e\| = 1$, we have $k = 1$. Thus $\|x\| = |\theta| < 1$, and the theorem follows.

That p non-zero coefficients suffice, is a consequence of Carathéodory's Theorem and a limit-argument, and it is easily checked that we can choose $a_1 = 0$.

References

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