NON UNIT DISTANCE GRAPHS

GUNDORPH K. KRISTIANSEN

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K_4.

I shall show that K_4 can be an induced subgraph of a unit distance graph only under special circumstances.

We consider a normed topology of the plane, and we can further assume that we have a polygonal geometry, i.e. the unit circle is a polygon.

The vertices of K_4 are denoted z_j with $j = 1, \ldots, 4$. The unit circle must contain the points $w_{j,k} = z_j - z_k$ for all $\{j, k\}$ with $j \neq k$.

Some simple conclusions can be drawn with respect to the numbers z_j in this case.

Assume that one vertex, z_j, belongs to the convex hull of the others. So,

$$z_j = \sum_{k \neq j} c_k z_k,$$

where the numbers c_k, $k \neq j$, are non-negative and have sum 1. For at least one m we have $0 < c_m < 1$.

But then

$$\sum_{k \neq j,m} c_k (z_k - z_m) / \sum_{k \neq j,m} c_k = (z_j - z_m)/(1 - c_m).$$

Taking norms we obtain

$$1 \geq 1/(1 - c_m),$$

a contradiction.

Next we shall use the fact that, for arbitrary $w \in \mathbb{C}$, the set $D_w = \{z \in \mathbb{C} \mid \|z - w\| \leq 1\}$ is convex.

We will need to use, in addition to the polygonal topology, also the usual topology of the complex plane. In the former, the length of a vector z is denoted by $\|z\|$, in the latter it is denoted by $|z|$.

If two vectors w_1 and w_2 are parallel, then $\|w_1\| = \|w_2\|$ implies $|w_1| = |w_2|$.

For notational simplicity the suffixes of vertices are understood modulo 4.

As follows from the results above, the convex hull of the four points z_1, z_2, z_3, z_4 has these four points as extreme points.

We may rename the points z_j and define the argument function such that for all $j \in [2, 4]$ we have

$$\arg w_{j,j-1} < \arg w_{j+1,j} < \arg w_{2,1} + 2\pi.$$

If we are still not satisfied, we may rename the points while maintaining their relative order (in other words, circular shifts are allowed).

With this in mind, the results obtained for a particular point (for instance z_1) are valid in general.

We introduce the auxiliary points $\zeta_1 = z_1 + z_3 - z_4$ and $\zeta_2 = z_2 + z_4 - z_3$.

Assume first that the line through z_1 and ζ_1 intersects the open interval (z_2, z_3) (in a point w_1, say).

If $\zeta_1 \in (z_1, w_1)$, we get a contradiction since D_{z_1} is convex.

If $w_1 \in (z_1, \zeta_1)$ we also get a contradiction, since now $z_1 \in D_{z_2}$.

Thus, $\zeta_1 = w_1$.

In the remaining case, the line through z_2 and ζ_2 intersects the open interval (z_1, z_4) (in a point w_2, say).

The argument proceeds as above, but with z_1 and z_2 interchanged.

We conclude that $\zeta_2 = w_2$.

In both cases we conclude that $z_1 - z_4$ and $z_2 - z_3$ have the same direction. Since they also have the same length, they must be equal.

Let now ζ_1 be an arbitrary point with distance 1 from z_1.

Assume that the interval (z_1, ζ_1) contains a point of (z_2, z_3).

Let now $\zeta_2 = z_2 + z_1 - \zeta_1$.

Because of the convexity of D_{z_2} we get a contradiction.

Thus, ζ_1 must belong to (z_2, z_3).

We conclude that the geometry is quadrangular.

Various sub graphs.

We continue our search for forbidden induced subgraphs in non-quadrangular geometries.

Let n be an integer greater then 3.

Let C_n ba a cycle with n edges.

Let q_n be a minimal integer with the property that the union of C_n with q_n particular chords cannot be an induced subgraph in a unit distance graph (unless the norm is further restricted).

We have essentially proved that $q_4 = 2$. We shall prove that $q_5 = q_6 = 3$, and that $q_n = 4$ for $n \geq 7$.

$n = 5$.

Without loss of generality we let $C_5 = z_1z_2z_3z_4z_5z_1$ with chords z_1z_3, z_1z_4, and z_2z_5.

We may assume that the unit distance graph G contains an induced triangle $z_1z_2z_3z_1$ in and, in fact, the set V of all points $m_1z_1 + m_2z_2 + m_3z_3$, where the integers m_1, m_2, m_3 have sum 1. Note that

$m_1z_1 + m_2z_2 + m_3z_3 = z_1 + m_2(z_2 - z_1) + m_3(z_3 - z_1)$,

so that each point in V can be obtained by starting from z_1 and proceeding by edges of length 1.

We shall try to locate z_4.

We first consider the interior of the convex hull of the neighbours of z_1 in V. Clearly z_4 cannot belong to this set.

Similarly for z_3.

Next, let w be an arbitrary neighbour of z_1 in V, and let w_1 and w_2 be the two common neighbours of z_1 and w in V.

Assume that z_4 belonged to the angular space between the two half-lines through w and directed away from w_1 and w_2, respectively.

Then the path $w_1 z_4 w_2$ consists of points at distance at least 1 from z_1. Now the line from z_1 through w intersects this path at a point w_3 different from w, which should then have a distance from z_1 strictly less than 1, and we have a contradiction.

We can repeat this argument for neighbours of z_3 in V.

A preliminary result:

Four particular angular spaces combine to two half-spaces, which results in the possible positions of z_4 to be restricted to a strip of breadth 2 parallel to the edge $z_1 z_3$.

Finally, the positions allowed for z_4 restrict to the union of the two intervals $[2z_1 - z_2, 2z_3 - z_2]$ and $[z_1 + z_2 - z_3, z_2 + z_3 - z_1]$.

Similarly, we find that z_5 must belong to the union of the two intervals

$[2z_1 - z_3, 2z_2 - z_3]$ and $[z_1 + z_3 - z_2, z_2 + z_3 - z_1]$.

Assume first that $z_4 \in (2z_1 - z_2, z_1 + z_3 - z_2)$.

Then the unit circle with centre z_1 has three points in common with the interval $[2z_1 - z_2, z_1 + z_3 - z_2]$ and so must contain it.

Then it also contains the interval $[z_1 + z_2 - z_3, z_2]$.

Further it follows that the unit circle with centre z_3 contains the intervals

$[z_2, z_2 + z_3 - z_1]$ and $[z_1 + z_3 - z_2, 2z_3 - z_2]$. There are similar results obtained by translation.

Similar arguments can be carried through for the z_5-intervals.

We will have to consider all possible combinations.

We first assume that z_4 belongs to one of the four open intervals considered above.

If $z_5 \in (z_1 + z_2 - z_3, 2z_2 - z_3]$, the interval $(z_1, z_5]$ intersects the interval

$(z_1 + z_2 - z_3, z_2)$ in a point with distance 1 from z_1, but z_5 also has distance 1 from z_1, and we have a contradiction.

A similar argument is valid if $z_5 \in [z_1 + z_3 - z_2, z_3)$.

We have a similar situation, if z_5 belongs to one of the relevant four open intervals:

We can then exclude the possibilities

$z_4 \in [z_1 + z_2 - z_3, z_2)$ and $z_4 \in (z_1 + z_3 - z_2, 2z_3 - z_2]$.

Next we consider simultaneous localizations of z_4 and z_5:

Let $z_4 \in (z_2, z_2 + z_3 - z_1)$, and $z_5 \in (z_3, z_2 + z_3 - z_1)$. Then the unit circle with centre in z_4 goes through z_3 and $z_4 + z_3 - z_1$, so that its interior contains z_5, a contradiction.

Let $z_4 \in (z_2, z_2 + z_3 - z_1)$, and $z_5 \in (2z_1 - z_3, z_1 + z_2 - z_3)$.

Then the unit circle with centre z_5 contains the three collinear points z_2, z_1, and $z_5 + z_3 - z_1$ and so the whole interval $[z_1, z_3]$.

But then the interval (z_5, z_4), which has length 1, intersects (z_1, z_2) at a point at distance 1 from z_5, a contradiction.

Let $z_4 \in (2z_1 - z_2, z_1 + z_3 - z_2)$, and $z_5 \in (z_3, z_2 + z_3 - z_1)$.

Then the unit circle with centre z_4 contains $[z_1, z_2)$, which is incompatible with $\|z_4 - z_5\|$ being 1.

Let $z_4 \in (2z_1 - z_2, z_1 + z_3 - z_2)$, and $z_5 \in (2z_1 - z_3, z_1 + z_2 - z_3)$.

Then the unit circle with centre z_5 contains z_2 and z_4, so that any point on $(z_2.z_4)$, in particular the point of intersection between (z_5, z_1) and (z_2, z_4), has distance at most 1 from z_5, contradicting $\|z_1 - z_5\| = 1$.

Finally, we must consider the possibility that z_4 and z_5 might belong to V.

Since the geometry is not quadrangular, we are left with the case $z_4 = z_1+z_3-z_2$ and $z_5 = z_1 + z_2 - z_3$. But then the distance between z_4 and z_5 would be 2 and not 1.

This concludes the proof of the conjecture for $n = 5$.

$n \geq 6$.

We shall prove that $q_n \leq 4$.

Let C_n be the union of the edges $z_j z_{j+1}$ $(j = 1, 2, \ldots n)$.

If we choose the four chords z_1z_3, z_1z_{n-1}, z_3z_{n-1}, and z_2z_n, we evidently have an induced subgraph as the one we showed to be impossible in the case $n = 5$.

In the sequel, we shall also use the following simple observation;

Let a_1 and a_2 be two points with distance 1. Let a_3 be a point with distance 1 from both of these two points. Put $a_4 = a_1 + a_2 - a_3$. Let a_5 be a further point with distance 1 from a_1 and a_2. Then the unit circle has a side parallel to (a_1, a_2). In this case, the geometry would have to be quadrangular, a situation we have excluded.

$n = 6$.

We choose the three chords z_1z_3, z_1z_5, and z_2z_6.

If the unit circle has no side parallel to z_1z_2, we have $z_6 = z_1 + z_2 - z_3$.

If the unit circle has no side parallel to z_2z_3, we have $z_5 = z_1 + z_6 - z_2$, and so $z_5 - z_1 = z_1 - z_3$.

But z_4 also has distance 1 from z_3 and z_5. So the unit circles with centres in z_3 and z_5 both contain z_1 and z_4, and so also the interval $[z_1, z_4]$, which is thus parallel to a side in the unit circle, q.e.d.

$n \geq 7$.

We must show that $q_n > 3$.

Thus, for any choice of three chords, we must be able to find a unit distance graph, containing the union G_n of the cycle C_n with these chords as an induced subgraph.

There are several sub cases:

1) The three chords have a common end point (z_1, say). Then we must find a path from z_2 to z_n containing the other end points of the chords.

These end points divide the mentioned $n-2$-path in four sub paths, at least one of these having length at least two. Such a sub path can easily be constructed without posing any restriction the other sub paths. The remaining sub paths each contains a single edge. Let $[z_j, z_{j+1}]$ (with j maximal) be such an edge. Assume that $[z_{j-1}, z_j]$ is an edge of the same type. Then, according to the observation above, we must have $zj - 1 = z_1 + z_j - z_{j+1}$. This may continue for at most one more interval, so that at most the points z_{j-1} , and z_{j-2} are determined by z_j and z_{j+1}.

2) The three chords form a triangle. Their end points divide C_n into three paths, each of length at least two. We easily supply the necessary vertices.

3) To construct worst case scenarios we must be able to use the above observation, which again requires a situation equivalent to the following:

The two points z_n and z_p (with $2 < p < n$) both have distances one from z_1 and z_2, and so we have $z_p = z_1 + z_2 - z_n$. There are three sub cases;

3a) $p = n - 1$. But this would make the geometry quadrangular, contrary to assumption.

3b) $3 < p < n - 1$. We easily locate the vertices on the two paths $[z_3, z_{p-1}]$ and $[z_{p+1}, z_{n-1}]$.

3c) $p = 3$. We must specify the third chord. This should be done in such a way that use of the observation above results in a further determination of a vertex as dependent on z_1, z_2, and z_n. There are two possibilities:

3c1) The chord is z_2z_4. We get $z_4 = z_2 + z_3 - z_1 = 2z_2 - z_n$. The remaining vertices on the path $[z_5, z_{n-1}]$ can easily be found.

3c2) The chord is z_1z_{n-1}. Then $z_{n-1} = z_1 - z_2 + z_n$. The vertices on the path $[z_4, z_{n-2}]$ are easy.

QED

The extended Moser graph.

Let G be the union of the Moser graph and an extra vertex w with edges connecting w with each vertex in the Moser graph. The graph G is not four colourable, and we shall find out, whether G can be an induced subgraph in a unit distance graph in a suitable topology.

However, the induced subgraph of G containing w and an arbitrary triangle of the Moser graph is a K_4. So the geometry must be quadrangular, and any unit distance graph is four colourable.

We cannot extend this kind of argument by replacing the Moser graph by an arbitrary four-chromatic graph, since, as follows from a well known theorem, proved by Blanche Descartes (actually William T. Tutte), there are four-chromatic graphs not containing any triangle.

On non-quadrangular geometries.

If G is four-chromatic, it cannot be perfect.

Consider the chromatic polynomial

$$P(G)(x) = \sum_{j=0}^{n} (-1)^{n-j} c_j(G) x^j,$$

where $n = |G|$ is the number of vertices of G, and the coefficients $c_j(G)$ are non-negative.

To determine the coefficients we use the following relationship

$$P(G) = P(G/E) - P(G \setminus E),$$

where E is an arbitrary edge in G, the graphs G/E and $G \setminus E$ are obtained from G by removing E, in the first case identifying its endpoints.

Let $\|G\|$ be the number of edges in the graph G.

Then we shall usually have $\|G/E\| = \|G\| - 1$, except when E is part of a triangle contained in G. In this case usually $\|G/E\| = \|G\| - 2$.

In fact, if E is part of more than one triangle, $\|G/E\|$ may be even smaller.

In the case where G is a unit distance graph, we cannot be sure that this property is inherited by G/E.

But in any case, we have the recursion,

$$c_j(G) = c_j(G/E) + c_j(G \setminus E),$$

valid for $j = 0, 1, \ldots, n-1$. We have

$$c_n(G) = c_n(G/E) = 1,$$

and

$$c_{n-1}(G) = 1 + c_{n-1}(G \setminus E) = \|G\|.$$

.

Notation: Let G_1 and G_2 be graphs with $|G_1 \cap G_2| = 1$. We then denote $G_1 \cup G_2$ by $G_1 \star G_2$.

More generally:

Let $G_1, \ldots, G_m$ be graphs satisfying $|G_j \cap G_k| = 1$ for $|j-k| = 1$ and $G_j \cap G_k = \emptyset$ for $|j-k| > 1$.

We then denote $\cup_{j=1}^m G_j$ by $G_1 \star G_2 \star \cdots \star G_m$.

Additionally, we can consider two graphs G_1 and G_2 with $|G_1 \cap G_2| = 2$. We then denote $G_1 \cup G_2$ by $G_1 \star\star G_2$.

Note that $P(G_1 \star\star G_2)$ may depend on parameters in addition to $|G_1|$ and $|G_2|$.

Exanples:

$$\begin{aligned}
P(T_n)(x) &= x(x-1)^{n-1}. \\
P(C_n)(x) &= (x-1)^n + (-1)^n(x-1). \\
P(T_n \star C_m)(x) &= (x-1)^n((x-1)^{m-1} + (-1)^m)). \\
P(C_n \star C_m)(x) &= ((x-1)^{n-1} + (-1^n))((x-1)^{m-1} + (-1)^m))(x-1)^2/x. \\
P(C_{m_1} \star \cdots \star C_{m_n})(x) &= \frac{1}{x^{n-1}} \prod_{j=1}^{n} P(C_{m_j})(x).
\end{aligned}$$

Let T_p and T_q intersect in two points, w_1 and w_2 Let p_1 and q_1 be the number of points in the open interval (w_1, w_2) belonging to T_p and T_q, respectively.

Then

$$P(T_p \star\star T_q)(x) = (x-1)^{p+q-p_1-q_1-3}((x-1)^{p_1+q_1+1} + (-1)^{p_1+q_1}).$$

Similarly, Let T_n and C_m intersect in two points, w_1 and w_2. Let m_1 and m_2 be the number of points in the two open intervals (w_1, w_2) belonging to C_m.

Then

$$\begin{aligned}
&P(T_n \star\star C_m)(x) = \\
&P(C_m)(x)P(C_n)(x)/x(x-1) + (-1)^{n-1}P(C_{m_1+1})(x)P(C_{m_2+1})(x)/x.
\end{aligned}$$

The natural next step is to investigate the unit distance graphs G containing edges E such that there is an isomorphism between G/E and a unit distance graph.